King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2005

Time allowed: THREE Hours

Candidates should answer ALL SIX parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

The following information defines terms used in this examination and may be of use.

• In a general curvilinear coordinate system (q_1, q_2, q_3) the unit base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ are given by

$$\mathbf{e}_i = \frac{1}{h_i}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right),$$

where $h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|$ are the corresponding scale factors.

• The *cylindrical coordinates* (r, θ, z) are defined by the transformation equations

$$x = r\cos\theta, \;\; y = r\sin\theta, \;\; z = z.$$

The Laplacian of a function Ψ in these coordinates is

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial z^2}$$

• The *spherical coordinates* (r, θ, ϕ) are defined by the transformation equations:

$$x = r\sin\theta\cos\phi, \;\; y = r\sin\theta\sin\phi, \;\; z = r\cos\theta$$

and the Laplacian of a function Ψ in these coordinates is

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

• Functions $\phi_n(x) = \sin nx$ and $\varphi_n(x) = \cos nx$ satisfy the following identities:

$$\int_0^{2\pi} \phi_n(x)\phi_m(x)\mathrm{d}x = \int_0^{2\pi} \varphi_n(x)\varphi_m(x)\mathrm{d}x = \pi\delta_{nm}$$

$$\int_0^{2\pi} \varphi_n(x)\phi_m(x)\mathrm{d}x = 0$$

where $n = 1, 2, 3, ...$

SECTION A – Answer SIX parts of this section

1.1) A system of curvilinear coordinates (α, β, γ) is given by the following relations:

$$\alpha = x + y, \;\; \beta = x - y, \;\; \gamma = z$$

Express the unit base vectors $\mathbf{e}_\alpha$, $\mathbf{e}_\beta$ and $\mathbf{e}_\gamma$ of this system in terms of the unit base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ of the Cartesian system, and check if the new coordinate system is orthogonal.

[7 marks]

1.2) Using the integral representation of the Dirac delta function,

$$\delta(x) = \int_{-\infty}^{\infty} e^{2\pi i \nu x} \mathrm{d}\nu,$$

evaluate the following double integral:

$$\int_{-\infty}^{\infty} H(x-1) \left[e^{-x} + \int_{-\infty}^{\infty} e^{-x^2 + 2\pi i x y} \mathrm{d}y \right] \mathrm{d}x,$$

where $H(x)$ is the Heaviside unit step function.

[7 marks]

1.3) Calculate the Fourier transform (FT) of the function

$$f(t) = t e^{-\alpha |t|}, \quad \alpha > 0$$

[7 marks]

1.4) Specify and classify the singular points of the differential equation

$$(x^2 - 4) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (x+2) \frac{\mathrm{d}y}{\mathrm{d}x} + (x-2) y = 0$$

[7 marks]

1.5) Show that the wave equation

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

in a spherically symmetrical case can be rewritten as follows:

$$\frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

where $\Psi = \Psi(r,t)$, v is a constant and r is the distance from the centre of symmetry. Using the method of separation of variables, obtain two ordinary differential equations for two functions, one which depends only on r and the other only on t.

[7 marks]

1.6) Give the definition of the Laplace transform (LT) $F(s) = \mathcal{L}\left[f(t)\right]$ of a function $f(t)$. Hence, calculate the LT of the function

$$f(t) = \sinh(\alpha t) = \frac{1}{2}\left(e^{\alpha t} - e^{-\alpha t}\right)$$

(you may assume that $\mathrm{Re}(s-\alpha) > 0$).

[7 marks]

SECTION B – Answer TWO questions

2) Consider the following differential equation

$$4x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + (x+3)y = 0$$

a) Find and classify all singular points of this equation.

[2 marks]

b) Using the generalised series expansion for the solution (the Frobenius method) around the $x = 0$ point,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+s},$$

show that the two solutions of the corresponding indicial equation for s can be chosen as $s_1 = \frac{1}{2}$ and $s_2 = \frac{3}{2}$, while the recurrence relation for the coefficients is:

$$a_{n+1} = \frac{1}{2(n+s)+1}a_n, \quad n = 0, 1, 2, ...$$

[13 marks]

c) Considering the coefficient a_0 as arbitary, derive the **four first terms** of two independent series solutions of the equation, $y_1(x)$ and $y_2(x)$.

[10 marks]

d) Why is it sufficient to keep only $x^{1/2}$ as $y_1(x)$ and ignore the rest of the series? [Hint: compare the rest of the series with $y_2(x)$.]

[3 marks]

e) Hence, state the general solution of the equation.

[2 marks]

3) Consider a thin circular plate of radius a. One semicircular boundary of it is held at a constant temperature of $100°$, while the other is kept at $0°$.

a) Explain why the heat transport equation

$$\nabla^2 T = \frac{1}{\mu^2}\frac{\partial T}{\partial t}$$

can in this case be rewritten as

$$\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0,$$

where $T = T(r,\theta)$ depends on the distance r from the centre of the plate and the polar angle θ.

[4 marks]

b) Write down the appropriate boundary conditions for this problem. What should we require the solution to be at the centre of the plate?

[2 marks]

c) Using the method of separation of variables, show that the functions $R(r)$ and $\Theta(\theta)$ of an elementary solution $R(r)\Theta(\theta)$ for $T(r,\theta)$ satisfy the following ordinary differential equations (ODE's):

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + k\Theta = 0 \text{ and } r^2\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - kR = 0,$$

where k is the corresponding separation constant.

[4 marks]

d) Show that the only choice for the constant k that ensures 2π periodicity of the function $\Theta(\theta)$ is $k = n^2$, where $n = 0, 1, 2, \ldots$. Hence, find a general solution of the ODE for $\Theta(\theta)$.

[4 marks]

e) Obtain two independent solutions of the ODE for $R(r)$ using a trial solution $R(r) \propto r^\alpha$. Explain why only one solution can be used for this problem. Hence, show that a general solution of the heat equation for the plate is

$$T(r,\theta) = \sum_{n=0}^{\infty} \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right] r^n$$

[4 marks]

f) Finally, using the boundary conditions at the rim of the plate, $r = a$, derive expressions for the unknown coefficients A_n and B_n. [Hint: use integration over the whole range $0 \le \theta < 2\pi$ and consider the case of the coefficients A_0 and B_0 separately.]

[12 marks]

4)

a) Calculate the Laplace transform (LT) $\mathcal{L}[f(t)]$ of the function $f(t) = e^{\mathrm{i}\alpha t}$ and show that

$$\mathcal{L}[\cos(\alpha t)] = \frac{s}{s^2+\alpha^2} \quad \text{and} \quad \mathcal{L}[\sin(\alpha t)] = \frac{\alpha}{s^2+\alpha^2}$$

[2 marks]

b) Prove the convolution theorem:

$$\mathcal{L}\left[\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau\right] = \mathcal{L}[f(t)]\,\mathcal{L}[g(t)]$$

[Hint: write the left-hand side as a double integral using the definition of the LT, and then change the order of integration.]

[8 marks]

c) Use the convolution theorem and the fact that $\mathcal{L}[1] = \frac{1}{s}$ to show that

$$\mathcal{L}^{-1}\left[\frac{\alpha^2}{s(s^2+\alpha^2)}\right] = 1 - \cos(\alpha t)$$

Show that the same result can also be obtained by calculating the LT of $f(t) = 1 - \cos(\alpha t)$ directly.

[7 marks]

d) Prove the following identity:

$$\mathcal{L}\left[\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right] = s\mathcal{L}[f(t)] - f(0)$$

[Hint: use the definition of the LT and integration by parts.]

[3 marks]

e) Using the above results, apply the LT method to solve the following system of first order differential equations

$$\frac{\mathrm{d}z}{\mathrm{d}t} + 2y = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}t} - 2z = 2$$

subject to the initial conditions $z(0) = y(0) = 0$.

[10 marks]