King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2004

Time allowed: THREE Hours

Candidates should answer no more than SIX parts of SECTION A, and no more than TWO questions from SECTION B.
No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

The following information defines terms used in this examination and may be of use.

• In a general curvilinear coordinate system (q_1, q_2, q_3) the unit base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$ are given by

$$\mathbf{e}_i = \frac{1}{h_i}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right),$$

where $h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|$ are the corresponding scale factors.

• The *cylindrical coordinates* (r, θ, z) are defined by the transformation equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

• The *spherical coordinates* (r, θ, ϕ) are defined by the transformation equations:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and the corresponding scale factors are $h_r = 1$, $h_\theta = r$ and $h_\phi = r\sin\theta$.

• The Laplacian of a function $\Psi(q_1, q_2, q_3)$ in general orthogonal curvilinear coordinates (q_1, q_2, q_3) is given by:

$$\nabla^2\Psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Psi}{\partial q_3}\right)\right]$$

where h_1, h_2 and h_3 are the corresponding scale factors.

• Functions $\phi_n(x) = \sin k_n x$ with $k_n = \frac{\pi n}{a}$ and $n = 1, 2, 3, ...$ are orthogonal:

$$\int_0^a \phi_n(x)\phi_m(x)\mathrm{d}x = \delta_{nm}\frac{a}{2}$$

If a function $f(x)$ is expanded in these functions, i.e. $f(x) = \sum_n f_n\phi_n(x)$, then the coefficients are:

$$f_n = \frac{2}{a}\int_0^a f(x)\phi_n(x)\mathrm{d}x$$

SECTION A – Answer SIX parts of this section

1.1) Show that the unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ for the *cylindrical coordinates* (r, θ, z) can be expressed via the Cartesian vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows:

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \;\; \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \;\; \mathbf{e}_z = \mathbf{k}$$

[7 marks]

1.2) Let $\Psi(q_1, q_2, q_3)$ be a scalar field defined in a general orthogonal curvilinear coordinate system (q_1, q_2, q_3). Show that the gradient is expressed via unit base vectors $\mathbf{e}_i$ and scale factors h_i $(i = 1, 2, 3)$ as

$$\text{grad}\Psi = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial \Psi}{\partial q_i}\mathbf{e}_i$$

[7 marks]

1.3) Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-2x}\left[\delta(2x+1) + 5H(x+1)\right]\mathrm{d}x$$

where $H(x)$ is the Heaviside unit step function.

[7 marks]

1.4) The integral Fourier transform, $F(\nu) = \mathcal{F}\left[f(t)\right]$, of a function $f(t)$ can be written as the integral:

$$F(\nu) = \int_{-\infty}^{\infty} f(t)e^{\mathrm{i}2\pi\nu t}\mathrm{d}t$$

Write down the inverse Fourier transform $f(t) = \mathcal{F}^{-1}\left[F(\nu)\right]$. What conditions should the function $f(t)$ satisfy for the Fourier transform to exist? Hence, explain why the Fourier transform does not formally exist for the Heaviside unit step function.

[7 marks]

1.5) Calculate the Fourier transform of the function $f(t)$ which is zero everywhere except for the interval $-1 \leq x \leq 1$ where it is equal to 1. Hence, using the inverse Fourier transform, express this function as an integral from $-\infty$ and ∞.

[7 marks]

1.6) Specify and classify the singular points of the differential equation

$$(x^2+1)(x^2-1)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3(x-1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2(x+1)^2y = 0$$

[7 marks]

1.7) Separate the variables in the heat transport equation

$$\frac{1}{\kappa}\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

Hence obtain two ordinary differential equations for the two functions in the corresponding elementary solution, one involving x and another t.

[7 marks]

1.8) Calculate the first three Legendre polynomials $P_n(x)$ $(n = 0, 1, 2)$ using the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n$$

Check that $P_2(x)$ is orthogonal to both $P_0(x)$ and $P_1(x)$.

[7 marks]

SECTION B – Answer TWO questions

2) Consider the *cylindrical coordinates* (r, θ, z). The unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ for this system can be expressed via the Cartesian vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as follows:

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \;\; \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \;\; \mathbf{e}_z = \mathbf{k}$$

a) Obtain Cartesian vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ via the unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$.

[5 marks]

b) The equations of motion of a point particle are given by $r = r(t)$, $\theta = \theta(t)$ and $z = z(t)$ (t is time). Show that the time derivatives of the unit base vectors are given by:

$$\frac{d\mathbf{e}_r}{dt} = \dot{\theta}\mathbf{e}_\theta, \;\; \frac{d\mathbf{e}_\theta}{dt} = -\dot{\theta}\mathbf{e}_r, \;\; \frac{d\mathbf{e}_z}{dt} = 0.$$

[7 marks]

c) By considering two close points A and B whose coordinates in a general curvilinear coordinate system are (q_1, q_2, q_3) and $(q_1 + \mathrm{d}q_1, q_2 + \mathrm{d}q_2, q_3 + \mathrm{d}q_3)$, show that the vector $\mathrm{d}\mathbf{r}$ connecting the two points in first order with respect to $\mathrm{d}q_i$ $(i = 1, 2, 3)$ is

$$\mathrm{d}\mathbf{r} = \sum_{i=1}^{3} h_i \mathrm{d}q_i \mathbf{e}_i,$$

where h_i is the scale factor.

[4 marks]

d) Using the results of the previous two questions, show that the velocity, $\mathbf{v}$, and acceleration, $\mathbf{a}$, of the particle are given by:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$$

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z$$

[8 marks]

e) Assuming that the particle is of unit mass and moves within the $z = 0$ plane in a central force field $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{e}_r$, find differential equations for both $r(t)$ and $\theta(t)$. (Hint: write down equations of motion along directions $\mathbf{e}_r$ and $\mathbf{e}_\theta$.) Hence, show that $\theta(t)$ will change linearly with time if the particle moves along a circular trajectory within the plane.

[6 marks]

3) The integral Fourier transform $F(\nu) = \mathcal{F}[f(t)]$ of a function $f(t)$ is defined as

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{\mathrm{i}2\pi\nu t} \mathrm{d}t$$

a) Find the Fourier transform, $\mathcal{F}[\delta(t)]$, of the Dirac delta function $\delta(t)$. Hence, prove the following integral representation for this function:

$$\delta(t) = \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi\nu t} \mathrm{d}\nu$$

[5 marks]

b) The convolution $f(t) * g(t)$ of two functions $f(t)$ and $g(t)$ is defined as an integral

$$p(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \mathrm{d}\tau$$

Prove the *convolution theorem* that the Fourier transform $P(\nu) = \mathcal{F}[p(t)]$ of $p(t)$ is equal to a product of Fourier transforms of the constituent functions, i.e. $P(\nu) = F(\nu)G(\nu)$.

[6 marks]

c) The function $f(t)$ is defined as $e^{-\alpha t}$ for $t \geq 0$ $(\alpha > 0)$ and zero otherwise. Show that the convolution of this function with itself $d(t) = f(t) * f(t) = t f(t)$.

[6 marks]

d) Show that the Fourier transform of the function $f(t)$ defined above is $F(\nu) = (\alpha - \mathrm{i}2\pi\nu)^{-1}$, while the Fourier transform of the function $d(t) = t f(t)$ is $D(\nu) = (\alpha - \mathrm{i}2\pi\nu)^{-2}$.

[8 marks]

e) Inversely, show, using the convolution theorem and the definitions of functions $f(t)$ and $d(t)$ given above, that the function whose Fourier transform is $D(\nu) = (\alpha - \mathrm{i}2\pi\nu)^{-2}$ is indeed $d(t)$.

[5 marks]

4) Consider the following differential equation

$$36x^2\frac{d^2y}{dx^2} + (5 - 9x^2)y = 0$$

a) Find and classify all singular points of this equation.

[2 marks]

b) Using the generalised series expansion for the solution (the Frobenius method),

$$y(x) = x^s \sum_{n=0}^{\infty} a_n x^n,$$

show that the two solutions of the corresponding indicial equation for s can be chosen as $s_1 = \frac{1}{6}$ and $s_2 = \frac{5}{6}$, while the recurrence relation for the coefficients a_n is:

$$a_n = \frac{9}{36(n+s)(n+s-1)+5} a_{n-2}, \quad n = 2, 3, ...$$

[12 marks]

c) Then, considering the coefficient a_0 as arbitary, derive three first terms of **two** independent series solutions of the equation, $y_1(x)$ and $y_2(x)$.

[14 marks]

d) Hence, state the general solution of the equation.

[2 marks]

5) Consider a metal sphere of radius a, initially at zero temperature, placed at $t = 0$ in a big water reservoir held at a constant temperature of 10 degrees.

a) Explain why the heat transport equation

$$\frac{1}{\mu^2}\frac{\partial u}{\partial t} = \Delta u$$

in this case can actually be written in a simplified form as

$$\frac{1}{\mu^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}$$

where the temperature $u = u(r,t)$ spatially depends only on the distance r from the sphere centre.

[5 marks]

b) Using a physical argument, write down the stationary (at $t \to \infty$) distribution $u_\infty(r) = u(r,\infty)$ of temperature in the sphere. Hence, write down a partial differential equation and the corresponding boundary and initial conditions for a new function $v(r,t) = u(r,t) - u_\infty(r)$.

[4 marks]

c) Assuming a negative separation constant $-k^2$, show that the method of separation of variables for the function $v(r,t)$ results in the following two ordinary differential equations (ODEs)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -(\mu k)^2 T \;\text{ and }\; \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + k^2 R = 0$$

for the two functions $T(t)$ and $R(r)$ to be introduced which depend on t and r, respectively.

[5 marks]

d) Check that the functions

$$T(t) = e^{-(\mu k)^2 t} \;\text{ and }\; R(r) = \frac{\sin kr}{r}$$

satisfy the ODEs above. Explain why the separation constant was chosen negative.

[4 marks]

e) Apply the boundary conditions on the sphere surface and deduce that the constant k can only take the following discrete values: $k_n = \frac{\pi n}{a}$, $n = 1, 2, 3, \ldots$.

[4 marks]

f) Hence, show that a general solution of the heat transport equation for the sphere is

$$v(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} v_n e^{-(\mu k_n)^2 t} \sin k_n r$$

[2 marks]

g) Finally, apply the initial conditions to find the unknown coefficients v_n and hence give the complete solution for $u(r,t)$.

[6 marks]