

King's College London

UNIVERSITY OF LONDON

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B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2003

Time allowed: THREE Hours

**Candidates must answer SIX parts of SECTION A,
and TWO questions from SECTION B.**

The approximate mark for each part of a question is indicated in square brackets.

**You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.**

**TURN OVER WHEN INSTRUCTED
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SECTION A – Answer SIX parts of this section

- 1.1) A general curvilinear orthogonal coordinate system (q_1, q_2, q_3) is given by the transformation functions:

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3).$$

Show that the unit base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the scale factors h_1, h_2, h_3 in this system are given, respectively, by the following relations:

$$\mathbf{e}_i = \frac{1}{h_i} \left(\frac{\partial \mathbf{r}}{\partial q_i} \right), \quad h_i = \left| \frac{\partial \mathbf{r}}{\partial q_i} \right|.$$

[7 marks]

- 1.2) Consider the spherical polar coordinates (r, θ, ϕ) ,

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Determine the scale factors h_r, h_θ, h_ϕ for this system and the unit base vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$.

[7 marks]

- 1.3) Evaluate the integral

$$\int_{-\infty}^{\infty} \delta \left(\frac{x}{5} - 1 \right) f(x) dx$$

where $f(x)$ is any continuous function.

[7 marks]

- 1.4) Give a formal definition of the integral Fourier transform, $F(\nu) = \mathcal{F}[f(t)]$, and its inverse, $f(t) = \mathcal{F}^{-1}[F(\nu)]$, for a function $f(t)$ defined in the interval $-\infty < t < \infty$. Hence derive the following integral representation for the delta function:

$$\delta(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t} d\nu$$

[7 marks]

1.5) Show that the Fourier transform of the function $f(t) = e^{-\alpha|t|}$ is

$$\mathcal{F}[f(t)] = \frac{2\alpha}{\alpha^2 + (2\pi\nu)^2}$$

Hint: when calculating the transform, split the integral over t into two: one with $t \leq 0$ and another with $t \geq 0$.

[7 marks]

1.6) Specify and classify the singular points of the differential equation

$$(x^2 - 1)(x - 4) \frac{d^2y}{dx^2} + (x - 1) \frac{dy}{dx} + (x + 1)y = 0$$

[7 marks]

1.7) Verify that the function

$$y(x, t) = y_1(x + ct) + y_2(x - ct)$$

with arbitrary functions $y_1(x)$ and $y_2(x)$ is a solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

[7 marks]

1.8) Use the generating function

$$G(x, t) = \frac{1}{\sqrt{1 - 2xt + t^2}}$$

for the Legendre polynomials $P_n(x)$ to derive explicit expressions for the polynomials with $n = 0, 1$.

[7 marks]

SECTION B – Answer TWO questions

- 2) The *parabolic coordinates* (u, v, θ) are specified by the following transformation equations:

$$x = uv \cos \theta, \quad y = uv \sin \theta, \quad z = \frac{1}{2} (u^2 - v^2),$$

where $u \geq 0$, $v \geq 0$ and $0 \leq \theta < 2\pi$.

- a) Show that the scale factors h_u , h_v , h_θ have the form:

$$h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\theta = uv$$

[9 marks]

- b) Show that the unit base vectors \mathbf{e}_u , \mathbf{e}_v , \mathbf{e}_θ can be expressed via the Cartesian vectors \mathbf{i} , \mathbf{j} , \mathbf{k} as follows:

$$\mathbf{e}_u = \frac{v}{\sqrt{u^2 + v^2}} \left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \frac{u}{v} \mathbf{k} \right)$$

$$\mathbf{e}_v = \frac{u}{\sqrt{u^2 + v^2}} \left(\cos \theta \mathbf{i} + \sin \theta \mathbf{j} - \frac{v}{u} \mathbf{k} \right)$$

$$\mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$$

[9 marks]

- c) Show that the parabolic coordinate system is orthogonal.

[4 marks]

- d) Given that the Laplacian operator ∇^2 of a scalar field $\Psi(x, y, z)$ in the general orthogonal curvilinear coordinates (q_1, q_2, q_3) is

$$\nabla^2 \Psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \Psi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \Psi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \Psi}{\partial q_3} \right) \right]$$

show that the Laplace equation $\nabla^2 \Psi(x, y, z) = 0$ can be rewritten in the parabolic coordinates in the following form:

$$\frac{1}{u^2 + v^2} \left[v \frac{\partial}{\partial u} \left(u \frac{\partial \Psi}{\partial u} \right) + u \frac{\partial}{\partial v} \left(v \frac{\partial \Psi}{\partial v} \right) \right] + \frac{1}{uv} \frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

[8 marks]

3) Let $F(\nu) = \mathcal{F}[f(t)]$ be the Fourier transform of a function $f(t)$.

a) Prove the following identity which is sometimes called the *modulation theorem*:

$$\mathcal{F}[f(t) \cos(2\pi\nu_0 t)] = \frac{1}{2} [F(\nu + \nu_0) + F(\nu - \nu_0)].$$

[6 marks]

b) Express the Fourier transforms of $f'(t) = \frac{df}{dt}$ and $f''(t) = \frac{d^2f}{dt^2}$ in terms of $F(\nu)$.
Hint: when calculating the Fourier transforms, use integration by parts.

[10 marks]

c) The n -th moment of a function $f(t)$ is given by the expression:

$$\mu_n = \int_{-\infty}^{\infty} t^n f(t) dt$$

By considering derivatives of the Fourier transform $F(\nu)$, show that

$$\mu_n = \frac{F^{(n)}(0)}{(-2\pi i)^n},$$

where $F^{(n)}(0)$ is the n -th derivative of $F(\nu)$ calculated at $\nu = 0$.

[8 marks]

d) Calculate the Fourier transform of the function $f(t)$ which is equal to unity if $0 \leq t \leq 1$ and zero otherwise.

[6 marks]

4) Consider the *Bessel differential equation*

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2)y = 0$$

a) Classify the singular points of this equation.

[4 marks]

b) Assuming that the parameter p in the Bessel equation is either noninteger or a negative number, use the Frobenius method to derive three first terms of **two** independent series solutions of the equation, $y_1(x)$ and $y_2(x)$.

[24 marks]

c) Hence, state the general solution of the equation.

[2 marks]

5) Consider the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$$

for a string of length L fixed at both ends.

a) Use the Fourier method of separation of variables to obtain two ordinary differential equations: one involving t and another x .

[6 marks]

b) Given that the separation constant $k = -p_n^2$, where $p_n = \frac{\pi}{L}n$ and $n = 1, 2, 3, \dots$, check that the appropriate solution of the equation involving x which is consistent with the boundary conditions is $\psi_n(x) = \sin(p_n x)$.

[4 marks]

c) Check that

$$\chi_n(t) = A_n \sin(cp_n t) + B_n \cos(cp_n t)$$

is a solution of the equation involving t , where A_n and B_n are arbitrary constants.

[3 marks]

d) Now assume that initially (i.e. at $t = 0$) the string is pulled by 0.06 units at $x = L/5$ and then released, i.e. the initial conditions are:

$$y(x, 0) = 0.3 \frac{x}{L}$$

for $0 \leq x \leq L/5$ and

$$y(x, 0) = 0.075 \left(1 - \frac{x}{L}\right)$$

for $L/5 \leq x \leq L$ and also $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$. Given that the general solution of the wave equation is

$$y(x, t) = \sum_{n=1}^{\infty} \psi_n(x) \chi_n(t),$$

determine the corresponding partial solution of the wave equation.

Hint: the integral $\int x \sin(p_n x) dx$ is calculated by parts.

[17 marks]

You may assume that functions $\psi_n(x)$ satisfy the orthogonality relation:

$$\int_0^L \psi_n(x) \psi_{n'}(x) dx = \frac{L}{2} \delta_{nn'}$$