King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 MATHEMATICAL METHODS IN PHYSICS II

Summer 1999

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) The circular cylindrical coordinates (ρ, ϕ, z) are defined by the transformation equations

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z\,,$$

where $0 \le \rho < \infty$, $0 \le \phi < 2\pi$ and $-\infty < z < \infty$. Determine the unit base vectors for this system.

[7 marks]

1.2) Show that, for a general curvilinear orthogonal coordinate system (q_1, q_2, q_3), the gradient of a scalar field $\psi(q_1, q_2, q_3)$ is given by

$$\text{grad}\,\psi = \sum_{i=1}^{3} \frac{\mathbf{e}_i}{h_i}\frac{\partial\psi}{\partial q_i}\,,$$

where $\{\mathbf{e}_i\,;i = 1, 2, 3\}$ and $\{h_i\,;i = 1, 2, 3\}$ denote the sets of unit base vectors and scale factors respectively for the coordinate system.

[7 marks]

1.3) State the general *filtering theorem* for the Dirac delta function $\delta(x)$. Hence evaluate the integral

$$\int_{-\infty}^{\infty} \delta\left(\frac{3t+2}{4}\right) \exp(-t^2)\,dt\,.$$

[7 marks]

1.4) Define the *Fourier transform* $\mathcal{F}[f(t)]$ of a function $f(t)$ which is defined on the interval $-\infty < t < \infty$. Calculate the Fourier transform of the Dirac delta function $\delta(t)$. Hence determine a formal integral representation for $\delta(t)$.

[7 marks]

1.5) Define the *convolution* $f * g$ of two functions $f(t)$ and $g(t)$. Prove that $f * g = g * f$.

[7 marks]

1.6) Explain what is meant by a *regular singular point* of a linear differential equation of second order. Classify all the singular points of the differential equation

$$x(x+1)^2\,\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + (x+3)\,\frac{\mathrm{d}y}{\mathrm{d}x} + (x-1)\,y = 0\,.$$

[7 marks]

1.7) Determine the *general* solution $R(r)$ of the differential equation

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - n(n+1)R = 0\,,$$

by using the trial solution $R(r) = r^s$, where $n = 0, 1, 2, \ldots$ and s is a constant.

[7 marks]

1.8) The general *axially symmetric* solution of the Laplace equation $\nabla^2 \psi = 0$ in spherical polar coordinates (r, θ, ϕ) is given by

$$\psi(r, \theta, \phi) = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1} \right) \, P_n(\cos\theta)\,,$$

where A_n and B_n are arbitrary contants and $P_n(\mu)$ denotes a Legendre polynomial. Use this result to determine the particular solution $\psi(r, \theta, \phi)$ of the Laplace equation which is *finite* in the region $0 \leq r \leq a$, and satisfies the boundary condition

$$\psi(a, \theta, \phi) = \cos\theta\,,$$

on the surface of the sphere $r = a$.

[7 marks]

[It may be assumed that $P_0(\mu) = 1$ and $P_1(\mu) = \mu$.]

SECTION B – Answer TWO questions

2) Define the *unit base vectors* $\{\mathbf{e}_i; i = 1, 2, 3\}$ and the *scale factors* $\{h_i; i = 1, 2, 3\}$ for a general three-dimensional curvilinear coordinate system.

[6 marks]

A particular curvilinear orthogonal coordinate system (q_1, q_2, q_3) is defined by the transformation equations

$$x = \cosh q_1 \cos q_2 \,,$$
$$y = \sinh q_1 \sin q_2 \,,$$
$$z = q_3,$$

where $0 \leq q_1 < \infty$, $0 \leq q_2 < 2\pi$ and $-\infty < q_3 < \infty$. Determine the unit base vectors $\{\mathbf{e}_i; i = 1, 2, 3\}$ and the scale factors $\{h_i; i = 1, 2, 3\}$ for this coordinate system, and prove that

$$h_1 = h_2 = (\sinh^2 q_1 + \sin^2 q_2)^{1/2}.$$

[16 marks]

Hence show that the Laplace equation $\nabla^2 \psi = 0$ can be expressed in the form

$$\frac{1}{(\sinh^2 q_1 + \sin^2 q_2)} \left(\frac{\partial^2 \psi}{\partial {q_1}^2} + \frac{\partial^2 \psi}{\partial {q_2}^2} \right) + \frac{\partial^2 \psi}{\partial {q_3}^2} = 0 \,.$$

[8 marks]

It may be assumed that the divergence of a vector field $\mathbf{F}$ in general curvilinear orthogonal coordinates is given by

$$\mathrm{div}\mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_3 h_1) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2) \right],$$

and the gradient of a scalar field $\psi(q_1, q_2, q_3)$ is given by

$$\mathrm{grad}\, \psi = \sum_{i=1}^{3} \frac{\mathbf{e}_i}{h_i} \frac{\partial \psi}{\partial q_i} \,.$$

3) Show that the Fourier transform $\mathcal{F}[f(t)] = F(\nu)$ of an **odd** function $f(t)$ can be written in the form

$$\mathcal{F}[f(t)] = -2\,\mathrm{i} \int_0^\infty f(t)\, \sin(2\pi\nu t)\, \mathrm{d}t\,.$$

[7 marks]

Prove that the Fourier transform $F(\nu)$ of the function

$$\begin{aligned} f(t) &= t \quad \text{for} \quad -1 \le t \le 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

is given by

$$F(\nu) = -\frac{\mathrm{i}}{2\pi^2\nu^2}\left[\sin(2\pi\nu) - (2\pi\nu)\cos(2\pi\nu)\right]\,.$$

[12 marks]

Use the inverse Fourier transform to evaluate the integral

$$\int_0^\infty \frac{\sin x}{x^2}\,(\sin x - x\cos x)\;\mathrm{d}x\,.$$

[11 marks]

4) Use the method of Frobenius to derive **two** independent series solutions of the differential equation

$$3x\,\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

in powers of x.

[24 marks]

Show that the series solutions converge for all $|x| < \infty$.

[6 marks]

5) Use the generating function for Legendre polynomials

$$(1-2\mu t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu)t^n$$

where $-1 \leq \mu \leq 1$ and $|t| < 1$, to prove the following results

(a) $P_n(1) = 1$,

[6 marks]

(b) $(2n+1)\mu P_n(\mu) = (n+1)P_{n+1}(\mu) + nP_{n-1}(\mu)$.

[12 marks]

Use the recurrence relation (b) to write $\mu P_{n+1}(\mu)$ in terms of $P_{n+2}(\mu)$ and $P_n(\mu)$, and also to write $\mu P_{n-1}(\mu)$ in terms of $P_n(\mu)$ and $P_{n-2}(\mu)$. Hence evaluate the integral

$$\int_{-1}^{1} \mu^2 P_{n+1}(\mu)P_{n-1}(\mu)\, d\mu\,.$$

[12 marks]

[It may be assumed that

$$\int_{-1}^{1} P_n(\mu)P_{n'}(\mu)\, d\mu = \frac{2}{2n+1}\delta_{n,n'}\,,$$

where $\delta_{n,n'}$ is the Kronecker delta function.]