King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2003

Time allowed: THREE Hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Show that, for a general curvilinear orthogonal coordinate system (q_1, q_2, q_3), the gradient of a scalar field $\psi(q_1, q_2, q_3)$ is given by

$$\text{grad}\,\psi = \sum_{i=1}^{3} \frac{\mathbf{e}_i}{h_i} \frac{\partial \psi}{\partial q_i}\,,$$

where $\{\mathbf{e}_i\,; i = 1, 2, 3\}$ and $\{h_i\,; i = 1, 2, 3\}$ denote the sets of unit base vectors and scale factors respectively for the coordinate system.

[7 marks]

1.2) Consider the spherical polar coordinates (r, θ, ϕ),

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Determine the scale factors h_r, h_θ, h_ϕ for this system and the unit base vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$.

[7 marks]

1.3) State the general *filtering theorem* for the Dirac delta function $\delta(x)$. Hence evaluate the integral

$$\int_{-\infty}^{\infty} \delta(4t + \pi)\sin(2t)\,dt\,.$$

[7 marks]

1.4) Define the *Fourier transform* $\mathcal{F}[f(t)]$ of a function $f(t)$ which is defined on the interval $-\infty < t < \infty$. Calculate the Fourier transform of the function

$$f(t) = H(t)\exp(-4\pi t)\,,$$

where

$$\begin{aligned} H(t) =& 0\,, \quad t < 0 \\ =& 1\,, \quad t \geq 0 \end{aligned}$$

is the Heaviside step function.

[7 marks]

1.5) The function $\phi(r, \theta, t)$ satisfies the partial differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = D\frac{\partial \phi}{\partial t} ,$$

where D is a constant. Separate the equation into three ordinary differential equations.

[7 marks]

1.6) Explain what is meant by a *regular singular point* of a linear differential equation of second order. Classify all the singular points of the differential equation

$$x^3(x-1)\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + x(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + (x-2)y = 0 .$$

[7 marks]

1.7) Verify that the function

$$y(x,t) = y_1(x+ct) + y_2(x-ct)$$

with arbitrary functions $y_1(x)$ and $y_2(x)$ is a solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

[7 marks]

1.8) Use the generating function for Legendre polynomials

$$(1 - 2\mu t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu)t^n ,$$

where $-1 \le \mu \le 1$ and $|t| \le 1$, to prove that

$$P_n(-\mu) = (-1)^n P_n(\mu)$$

for all $n = 0, 1, 2, \ldots$.

[7 marks]

SECTION B – Answer TWO questions

2) Show that the Fourier transform $\mathcal{F}[f(t)] = F(\nu)$ of an **even** function $f(t)$ can be written in the form

$$\mathcal{F}[f(t)] = 2\int_0^\infty f(t)\,\cos(2\pi\nu t)\,\mathrm{d}t\,.$$

[7 marks]

Prove that the Fourier transform $F(\nu)$ of the function

$$\begin{aligned} f(t) =& 1 - |t| \quad \text{for} \quad 0 \le |t| < 1 \\ =& 0\,, \qquad\quad \text{otherwise} \end{aligned}$$

is given by

$$F(\nu) = \left[\frac{\sin(\pi\nu)}{\pi\nu}\right]^2 .$$

[14 marks]

Use the inverse Fourier transform to evaluate the integral

$$\int_0^\infty \cos x \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x\,.$$

[9 marks]

3) Show that, for a general curvilinear orthogonal coordinate system (q_1, q_2, q_3), the gradient of a scalar field $\psi(q_1, q_2, q_3)$ can be written as

$$\text{grad}\,\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial q_3}\,,$$

where $\{h_i; i = 1,2,3\}$ and $\{\mathbf{e}_i; i = 1,2,3\}$ denote the sets of scale factors and unit base vectors respectively for the coordinate system.

[10 marks]

A particular curvilinear orthogonal coordinate system (q_1, q_2, q_3) is defined by the transformation equations

$$\begin{aligned} x &= q_1 q_2 \cos q_3\,, \\ y &= q_1 q_2 \sin q_3\,, \\ z &= \tfrac{1}{2}(q_1^2 - q_2^2)\,, \end{aligned}$$

where $q_1 \geq 0$, $q_2 \geq 0$ and $0 \leq q_3 < 2\pi$. Determine the scale factors $\{h_i; i = 1,2,3\}$ and unit base vectors $\{\mathbf{e}_i; i = 1,2,3\}$ for this system.

[12 marks]

Hence calculate the gradient of the scalar field

$$\psi(q_1, q_2, q_3) = (q_1^2 + q_2^2)\cos q_3$$

at the point P which has curvilinear coordinates $q_1 = 1$, $q_2 = 1$ and $q_3 = \frac{\pi}{4}$. Express your answer in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

[8 marks]

4) Consider the *Bessel differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

a) Classify the singular points of this equation.

[4 marks]

b) Assuming that the parameter p in the Bessel equation is either noninteger or a negative number, use the Frobenius method to derive three first terms of **two** independent series solutions of the equation, $y_1(x)$ and $y_2(x)$.

[24 marks]

c) Hence, state the general solution of the equation.

[2 marks]

5) Apply the method of separation of variables to the Laplace equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\,\phi^2}=0\,,$$

where $\psi=\psi(r,\theta,\phi)$ and (r,θ,ϕ) are spherical polar coordinates. Hence show that the physically acceptable **product** solutions of the Laplace equation, which are axially symmetric about the z axis and finite at the origin $r=0$, are given by

$$\psi(r,\theta,\phi)=r^nP_n(\cos\theta)\,,$$

where $n=0,1,2,\ldots$, and $P_n(\mu)$ denotes a Legendre polynomial in the variable $\mu=\cos\theta$.

[It may be assumed that the differential equation

$$\sin\theta\frac{d}{d\,\theta}\left(\sin\theta\frac{d\Theta}{d\,\theta}\right)+\left(U\sin^2\theta\right)\Theta=0\,,$$

only has a physically acceptable solution when the separation constant $U=n(n+1)$ with $n=0,1,2,\ldots$, and that this solution is given by the Legendre polynomial $P_n(\mu)$ with $\mu=\cos\theta$.]

[20 marks]

Determine the particular solution $\psi=\psi(r,\theta,\phi)$ of the Laplace equation which is single-valued and finite in the region $0\le r\le a$, and satisfies the boundary condition

$$\psi(a,\theta,\phi)=\cos^2\theta\,,$$

on the surface of the sphere $r=a$.

[Note that the first three Legendre polynomials are $P_0(\mu)=1$, $P_1(\mu)=\mu$ and $P_2(\mu)=\frac{1}{2}(3\mu^2-1)$.]

[10 marks]