King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

BSc EXAMINATION

CP/2260 Mathematical Methods in Physics II

SUMMER 2002

Time allowed: **THREE** HOURS

Candidates must answer any SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each question or part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A — answer any SIX parts of this section

1.1 Evaluate the integral

$$\int_{-\infty}^{\infty} (3\delta(3t-2) + 3H(t-2))e^{-3t}dt\,,$$

where $\delta(t)$ denotes the Dirac delta function and $H(t)$ denotes the Heaviside step function.

[7 marks]

1.2 A curvilinear coordinate system (q_1, q_2) is defined by the transformation equations

$$x = q_1 q_2, \quad y = (q_1^2 - q_2^2)/2\,.$$

Determine the unit base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ for this coordinate system. Are $\mathbf{e}_1$ and $\mathbf{e}_2$ orthogonal?

[7 marks]

1.3 Determine the Fourier transform of the function

$$f(x) = (\delta(x-a) + \delta(x+a))/2,$$

where $\delta(x)$ denotes the Dirac delta function and a is a constant.

[7 marks]

1.4 By assuming a solution of the form $y = Ax^c$, where A and c are constants, determine the general solution of the differential equation

$$x\frac{d}{dx}x\frac{dy}{dx} - n^2 y = 0\,,$$

where $n = 0, 1, 2, \ldots$.

[7 marks]

1.5 A possible solution of the two-dimensional wave equation in polar coordinates can be written in the form

$$\phi(r,t) = J_0(\omega r)(C\cos(\omega ct) + D\sin(\omega ct))\,,$$

where r is the radial distance, ω is a separation constant, c is the wave velocity, C and D are constants, and $J_0(x)$ is a Bessel function. Determine the general solution which satisfies the boundary condition that $\phi(R,t) = 0$ at all times $t \geq 0$.

[7 marks]

1.6 The function $\phi(r,\theta)$ satisfies the equation,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0\,.$$

Separate the equation into two ordinary differential equations involving a separation constant k. What are the allowed values of k if $k > 0$ and the solution $\phi(r,\theta)$ must be single valued as a function of θ?

[7 marks]

1.7 The generating function for Chebyshev polynomials $T_n(x)$ is

$$G(x,t) = \frac{1-xt}{1-2xt+t^2} = \sum_{n\geq 0} T_n(x)t^n \,,$$

where $|x| \leq 1$ and $0 \leq t < 1$. Determine the values of $T_n(0)$ for $n = 0, 1, 2, 3, 4$.

[7 marks]

1.8 The generating function for Legendre polynomials $P_n(x)$ is

$$G(x,t) = \sum_{n\geq 0} P_n(x)t^n = (1-2xt+t^2)^{-1/2}.$$

Show that

$$(x-t)\frac{\partial G}{\partial x} = t\frac{\partial G}{\partial t}\,.$$

Then show that

$$x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx} = nP_n\,.$$

[7 marks]

SECTION B – answer TWO questions

2. Classify the singular points of the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 9y = 0\,.$$

[Ignore the point at infinity.] What type of point is $x = 0$?

[8 marks]

Use the method of Frobenius to show that one solution of the equation is

$$y = a_0\left[1 - \frac{9}{2!}x^2 + \frac{45}{4!}x^4 + \frac{315}{6!}x^6 + \ldots\right],$$

and find the other solution.

[16 marks]

Show that the radius of convergence of the series solution is 1.

[6 marks]

3. Given that $g(\lambda)$ is the Fourier transform of $f(x)$ and that $g(\lambda)$ is an odd function of λ, show that

$$f(x) = 2i \int_0^{\infty} g(\lambda) \sin(2\pi\lambda x)\, d\lambda\,.$$

[6 marks]

Prove that the Fourier transform of the function

$$f(x) = \begin{cases} -1, & -1 < x < 0, \\ +1, & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}$$

is

$$g(\lambda) = -\frac{2i \sin^2 \pi\lambda}{\pi\lambda}\,.$$

[12 marks]

Use the inverse Fourier transform to evaluate the integral

$$\int_0^{\infty} \frac{\sin^3 t \cos t}{t} dt\,.$$

[12 marks]

4. The generating function for Legendre polynomials $P_n(x)$ is

$$G(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n\geq 0} P_n(x)t^n \,.$$

Deduce that

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0\,.$$

Hence deduce that

$$P_{n+2}(0) = -\frac{n+1}{n+2}P_n(0)\,.$$

[8 marks]

Show that $P_n(-x) = (-1)^n P_n(x)$.

[4 marks]

A spherical metal shell is split into two halves by a thin insulating layer. The centre of the sphere is at the origin and its radius is R. The hemisphere with $z > 0$ is at potential $+V$ and the hemisphere with $z < 0$ is at potential $-V$. In the region outside the sphere, that is when the radial distance $r \geq R$, the electrostatic potential $\phi(r,\theta)$ satisfies Laplace's equation and has the solution

$$\phi(r,\theta) = \sum_{n\geq 0}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)\,,$$

where θ is the polar angle, A_n and B_n are constants, and $P_n(\cos\theta)$ are Legendre polynomials.

What is the solution that satisfies the boundary condition $\phi(r,\theta) \to 0$ as $r \to \infty$?

[5 marks]

The orthogonality relation for Legendre polynomials is

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm}\,,$$

where δ_{nm} is the Kronecker delta. Use this relation and the boundary condition at the surface of the sphere that

$$\phi(R,\theta) = \begin{cases} +V, & 0 \leq \theta < \pi/2, \\ -V, & \pi/2 < \theta \leq \pi, \end{cases}$$

to show that $B_m = 0$ when m is even and

$$B_m = (2m+1)R^{m+1}V\int_0^1 P_m(x)dx$$

when m is odd.

[15 marks]

5. The temperature of a spherical object of radius R is given by the function $\phi(r,t)$ where t is the time and r is the radial distance from the centre of the sphere. The temperature obeys the heat conduction equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\phi}{\partial r} = \frac{1}{\alpha^2}\frac{\partial\phi}{\partial t},$$

where α is the coefficient of heat conduction.

Use the method of separation of variables to obtain two ordinary differential equations.

[4 marks]

By substitution check that

$$\psi(r) = \frac{1}{\omega r}[A\cos\omega r + B\sin\omega r],$$

is a solution of the equation involving r, where ω^2 is a separation constant. Then deduce that the general solution for $\phi(r,t)$ which is finite for all t is

$$\phi(r,t) = C - \frac{D}{r} + \frac{1}{\omega r}[A\cos\omega r + B\sin\omega r]\exp(-\alpha^2\omega^2 t),$$

where A, B, C, D are constants.

[10 marks]

The object is initially at temperature ϕ_0 and is placed in a bath of water at temperature ϕ_1. Show that the solution which satisfies the boundary conditions that $\phi(r,0) = \phi_0$ for $0 \leq r < R$, and $\phi(R,t) = \phi_1$ for all $t \geq 0$, is

$$\phi(r,t) = \phi_1 + (\phi_0 - \phi_1)\sum_{n\geq 1} b_n \frac{\sin(n\pi r/R)}{(n\pi r/R)}\exp(-\alpha^2 n^2\pi^2 t/R^2),$$

where the b_n are constants.

[8 marks]

Explain in principle how the orthogonality relation

$$\frac{2}{R}\int_0^R \sin(n\pi r/R)\sin(m\pi r/R)dr = \delta_{mn},$$

and the boundary condition at $t = 0$ can be used to find the constants b_n. It is *not* expected that you evaluate the constants.

[8 marks]