King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2001

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

**You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.**

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Evaluate the integral

$$\int_{-\infty}^{\infty} (2\delta(2t+3) + 3H(t+2))e^{-t}dt\,,$$

where $\delta(t)$ denotes the Dirac delta function and $H(t)$ denotes the Heaviside step function.

[7 marks]

1.2) A curvilinear coordinate system (q_1, q_2) is defined by the transformation equations

$$x = a\cosh q_1 \cos q_2, \quad y = a \sinh q_1 \sin q_2\,,$$

where a is a constant. Determine the unit base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ for this coordinate system.

[7 marks]

1.3) Determine the Fourier transform of the function

$$f(x) = \delta(x - \pi/\omega)\cos(\omega x),$$

where $\delta(x)$ denotes the Dirac delta function and ω is a constant.

[7 marks]

1.4) By assuming a solution of the form $y = Ax^c$, determine the general solution of the differential equation

$$3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0\,.$$

[7 marks]

1.5) A solution of the one-dimensional diffusion equation can be written in the form

$$\phi(x,t) = Ax + B + e^{-\alpha^2\omega^2 t}(C\cos\omega x + D\sin\omega x)\,.$$

Determine the general solution which satisfies the boundary conditions that $\phi(0,t) = \phi_0$ and $\phi(L,t) = \phi_1$, at all times $t \geq 0$, where ϕ_0 and ϕ_1 are constants.

[7 marks]

1.6) The function $\phi(x, y, z)$ satisfies Laplace's equation, $\nabla^2\phi = 0$, in cartesian coordinates. Separate the equation into three ordinary differential equations.

[7 marks]

1.7) The generating function for Legendre polynomials $P_n(x)$ is

$$G(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n\geq 0} P_n(x)t^n \,,$$

where $|x| \leq 1$ and $0 \leq t < 1$. Determine the values of $P_n(0)$ for $n = 0, 1, 2, 3, 4$.

[7 marks]

1.8) The generating function for Laguerre polynomials $L_n(x)$ is

$$G(x,t) = \sum_{n\geq 0} L_n(x)t^n = \frac{e^{-xt/(1-t)}}{1-t}.$$

Show that

$$\frac{dL_{n+1}}{dx} - \frac{dL_n}{dx} + L_n = 0\,.$$

[7 marks]

SECTION B – Answer TWO questions

2a) Use the series method to find a solution of the equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 4y = 0\,.$$

[8 marks]

b) Use the method of Frobenius to show that one solution of the equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0$$

is

$$y = a_0\left[1 - \frac{3}{2!}(2x^2) + \frac{3}{4!}(2x^2)^2 + \frac{3}{6!}(2x^2)^3 + \frac{9}{8!}(2x^2)^4 + \ldots\right],$$

and find the the other solution.

[18 marks]

Show that the series solution converges for all $|x| < \infty$.

[4 marks]

3) Show that the Fourier transform of an even function $f(x)$ can be written in the form

$$g(\lambda) = 2\int_0^\infty f(x)\cos(2\pi\lambda x)\,dx\,.$$

[5 marks]

Prove that the Fourier transform of the function

$$f(x) = \exp(-a|x|), \quad \text{for } -\infty < x < \infty$$

is given by

$$g(\lambda) = \frac{2a}{a^2 + 4\pi^2\lambda^2}\,.$$

[10 marks]

Use the inverse Fourier transform to evaluate the integrals

$$\int_0^\infty \frac{1}{1+t^2}dt \quad \text{and} \quad \int_0^\infty \frac{\cos t}{1+t^2}dt\,.$$

[15 marks]

4) Given the recursion relations for Legendre polynomials P_n and their derivatives P'_n

$$
\begin{aligned}
(n+1)P_n &= P'_{n+1} - xP'_n \\
nP_n &= xP'_n - P'_{n-1}
\end{aligned}
$$

show that

$$\int P_n(x)dx = \frac{1}{2n+1}\left(P_{n+1}(x) - P_{n-1}(x)\right) .$$

[5 marks]

Using the generating function for Legendre polynomials given in question 1.7 determine the values of $P_n(1)$ and $P_n(-1)$.

[6 marks]

Suppose that the function

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 1, & 0 < x \leq 1, \end{cases}$$

is expanded in a series of Legendre polynomials as

$$f(x) = \sum_{n \geq 0} a_n P_n(x) .$$

Show that $a_n = 0$ when n is even, and when n is odd,

$$a_n = -P_{n+1}(0) + P_{n-1}(0) .$$

[12 marks]

Determine the expansion of $f(x)$ in Legendre polynomials up to and including the term in P_3.

[7 marks]

[You are given that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm} .]$$

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5) The temperature at all points of a cylindrical object is given by the function $\phi(r, z, t)$ where t is the time. The cylinder has height L and radius R. The coordinate system is chosen so that the circular ends of the cylinder are in the planes $z = 0$ and $z = L$, and the centreline of the cylinder coincides with the z-axis. The temperature obeys the heat conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial \phi}{\partial t},$$

where α is the coefficient of heat conduction.

Use the method of separation of variables to obtain three ordinary differential equations, namely,

$$\begin{aligned} &\frac{d\psi_3(t)}{dt} + \omega^2\alpha^2\psi_3(t) = 0 \\ &\frac{d^2\psi_2(z)}{dz^2} + (\omega^2 - \lambda^2)\psi_2(z) = 0 \\ &\frac{1}{r}\frac{d}{dr} r \frac{d\psi_1(r)}{dr} + \lambda^2\psi_1(r) = 0, \end{aligned}$$

where ω^2 and λ^2 are separation constants.

[6 marks]

Use the method of Frobenius to show that the series solution for $\psi_1(r)$, when $\lambda \neq 0$, is

$$\psi_1(r) = a_0 \sum_{n\geq 0} (-1)^n \frac{(\lambda r/2)^{2n}}{(n!)^2},$$

where a_0 is a constant. This solution is proportional to the Bessel function $J_0(\lambda r)$.

[8 marks]

What is the solution for $\psi_1(r)$ when $\lambda = 0$?

[2 marks]

Assume that the solution for the temperature $\phi(r, z, t)$ which is finite when $r = 0$ is

$$\phi(r, z, t) = A + e^{-\alpha^2\omega^2 t} J_0(\lambda r) \left[B\cos(\sqrt{\omega^2 - \lambda^2}z) + C\sin(\sqrt{\omega^2 - \lambda^2}z) \right],$$

where A, B and C are constants and $t \geq 0$. At time $t = 0$ the object is placed in an oven which is at temperature ϕ_1 so that for all times $t \geq 0$ the temperature of the surface of the object is ϕ_1. Show that this boundary condition implies that

$$A = \phi_1, \quad \lambda = \frac{j_s}{R} \text{ and } \omega^2 = \frac{n^2\pi^2}{L^2} + \frac{j_s^2}{R^2},$$

where j_s is the s^{th} zero of $J_0(x)$ and $n, s = 1, 2, \ldots$.

[10 marks]

Write down the most general solution for $\phi(r, z, t)$

[4 marks]