King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2260 Mathematical Methods in Physics II

Summer 2000

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Evaluate the integral

$$\int_{-\infty}^{\infty} te^{-t^2}\delta(t+3)dt\,,$$

and show that

$$\int_{-\infty}^{\infty} (2t+3)\delta(4t+1)dt = 5/8\,,$$

where $\delta(t)$ denotes the Dirac delta function.

[7 marks]

1.2) An orthogonal curvilinear coordinate system (q_1, q_2) is defined by the transformation equations

$$x = \frac{1}{2}(q_1^2 - q_2^2), \quad y = q_1 q_2\,.$$

Determine the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ for this coordinate system and show that $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal.

[7 marks]

1.3) Show that the Fourier transform of the function

$$f(x) = H(x-1)e^{-x},$$

where $H(x)$ denotes the Heaviside step function, is

$$g(\lambda) = \frac{e^{-(1+2\pi i\lambda)}}{1+2\pi i\lambda}\,.$$

[7 marks]

1.4) By assuming a solution of the form $\phi = Ar^c$, determine the general solution of the differential equation

$$\frac{d}{dr}r^2\frac{d\phi}{dr} - n(n+1)\phi = 0\,,$$

where n is a positive integer.

[7 marks]

1.5) Determine the general solution of the differential equation

$$\frac{d^2\phi}{dx^2} + \omega^2\phi = 0$$

which satisfies the boundary conditions that $\phi(0) = \phi(L) = 0$.

[7 marks]

1.6) The function $\phi(r,\theta,t)$ satisfies the partial differential equation

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = D\frac{\partial \phi}{\partial t}\,,$$

where D is a constant. Separate the equation into three ordinary differential equations.

[7 marks]

1.7) The generating function for Legendre polynomials $P_n(x)$ is

$$G(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n\geq 0} P_n(x)t^n\,,$$

where $|x| \leq 1$ and $0 \leq t < 1$. Deduce that

$$P_n(-1) = (-1)^n.$$

[7 marks]

1.8) The generating function for Hermite polynomials $H_n(x)$ is

$$G(x,t) = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}.$$

Show that

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)\,.$$

[7 marks]

SECTION B – Answer TWO questions

2) Classify the singular points of the differential equation

$$3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x-1)y = 0\,.$$

[6 marks]

Use the method of Frobenius to show that one solution of the equation is

$$y = a_0 x \left[1 + \sum_{n\geq 1} \frac{(-1)^n (2x)^n}{n! \prod_{m=1}^{n} (3m+4)} \right],$$

where a_0 is a constant, and find the other series solution.

[18 marks]

Show that the series solutions converge for all $|x| < \infty$.

[6 marks]

3) Show that the Fourier transform of an even function $f(x)$ can be written in the form

$$g(\lambda) = 2\int_0^\infty f(x)\cos(2\pi\lambda x)\,dx\,.$$

[5 marks]

Prove that the Fourier transform of the function

$$f(x) = \begin{cases} 1-|x|, & \text{for } 0 \leq |x| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is given by

$$g(\lambda) = \left(\frac{\sin \pi\lambda}{\pi\lambda} \right)^2 .$$

[15 marks]

Use the inverse Fourier transform to evaluate the integral

$$\int_0^\infty \frac{\cos t \sin^2 t}{t^2} dt\,.$$

[10 marks]

4) The generating function for Legendre polynomials is

$$G(x,t) = \sum_{n\geq 0} P_n(x)t^n = (1 - 2xt + t^2)^{-1/2},$$

where $|x| \leq 1$ and $|t| < 1$. Prove that

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0\,.$$

[7 marks]

If $P_0(x) = 1$ and $P_1(x) = x$, what are $P_2(x)$ and $P_3(x)$?

[7 marks]

Express $\cos\theta$ and $\cos^2\theta$ as functions of the Legendre polynomials $P_n(\cos\theta)$.

[6 marks]

The solution of Laplace's equation in spherical polar coordinates (r, θ, φ) in a problem with azimuthal symmetry can be written in the form

$$\phi(r,\theta) = \sum_{n\geq 0} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)\,.$$

Determine the solution for $\phi(r,\theta)$ in the region $r \geq R$ which satisfies the boundary conditions that (i) $\phi \to 0$ as $r \to \infty$ and (ii) on the surface of the sphere $r = R$, $\phi(R,\theta) = \cos\theta - 3\cos^2\theta$.

[10 marks]

5) In plane polar coordinates (r,θ) Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0\,.$$

By assuming that $\phi(r,\theta)=\psi_1(r)\psi_2(\theta)$, use the method of separation of variables to derive two ordinary differential equations for ψ_1 and ψ_2.

[6 marks]

The general solution is single valued as a function of θ, (that is, $\phi(r,\theta)=\phi(r,\theta+2\pi)$) and is *not* independent of θ. Show that

$$\phi(r,\theta)=\sum_{n\geq 1}\left(A_n r^n+B_n r^{-n}\right)\left(C_n\cos n\theta+D_n\sin n\theta\right)\,,$$

where A_n, B_n, C_n and D_n are constants.

[10 marks]

Show that the solution which satisfies the boundary conditions that $\phi=0$ at $r=0$, and that at $r=a$,

$$\phi(a,\theta)=\begin{cases}-V, & -\pi<\theta<0,\\ V, & 0<\theta<\pi,\end{cases}$$

is

$$\phi(r,\theta)=\frac{4V}{\pi}\left[\frac{r}{a}\sin\theta+\frac{1}{3}\left(\frac{r}{a}\right)^3\sin 3\theta+\frac{1}{5}\left(\frac{r}{a}\right)^5\sin 5\theta+\ldots\right].$$

[14 marks]