King's College London

UNIVERSITY OF LONDON

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B.Sc. EXAMINATION

CP/2250 MATHEMATICAL METHODS IN PHYSICS I

Summer 1999

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED
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SECTION A – Answer SIX parts of this section

1.1) Find the solution of the differential equation

$$x\frac{dy}{dx} + y^2 = 0\,,$$

which satisfies the boundary condition that $y = 1$ when $x = 1$.

[7 marks]

1.2) Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0\,,$$

which satisfies the boundary conditions that $y = 1$ and $dy/dx = -3$ when $x = 0$.

[7 marks]

1.3) Given that the scalar field $\phi = 1/r$ where $r = (x^2 + y^2 + z^2)^{1/2}$, calculate grad ϕ.

[7 marks]

1.4) Is the vector field $\mathbf{E} = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ irrotational or solenoidal or neither?

[7 marks]

1.5) Calculate the eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}.$$

[7 marks]

1.6) With respect to a linear transformation A define the terms Hermitian and unitary. Is the transformation

$$\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Hermitian or unitary?

[7 marks]

1.7) Given the vector field $\mathbf{E} = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$ calculate the line integral $\int_C \mathbf{E}.d\mathbf{r}$ where C is the arc of the circle $x^2 + y^2 = 1$ in the x, y-plane, from the point (1,0,0) to (0,1,0).

[7 marks]

1.8) The Fourier series representation of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1\,, \\ 0 & \text{if } 1 < x < 2\,, \end{cases}$$

is

$$F(f(x)) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin n\pi x}{n}\,.$$

Sketch the function $f(x)$ and find the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,.$$

[7 marks]

SECTION B – Answer TWO questions

2a) The behaviour of a damped simple harmonic oscillator is determined by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \omega_0^2 y = \cos \omega t \,,$$

where ω_0 is the natural angular frequency of the oscillator and ω is the driving angular frequency. Find the general solution of this equation applicable when $\omega_0 > 1$.

[12 marks]

Show that if $\omega = \omega_0$ and $t \gg 1$ then the motion of the oscillator is $\pi/2$ out of phase with the driving force, and its amplitude is $1/(2\omega_0)$.

[6 marks]

b) The behaviour of a particle of mass m moving in an infinite one-dimensional potential well with walls at $x = 0$ and $x = a$ is described by Schrödinger's equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \,,$$

where E is the kinetic energy of the particle and ψ is the wavefunction. Show that if $\psi = 0$ at $x = 0$ and $x = a$, then the allowed solutions of the equation have energies

$$E = \frac{h^2}{8m}\frac{n^2}{a^2} \qquad \text{where} \quad n = 1, 2, 3, \ldots \,.$$

[12 marks]

3) In a set of (idealised) chemical reactions involving three chemical species the rate of change of the number N_i of each species i is given by the set of coupled differential equations

$$\frac{dN_1}{dt} = N_1 + 2N_2 + 2N_3$$
$$\frac{dN_2}{dt} = 2N_1 + 3N_2$$
$$\frac{dN_3}{dt} = 2N_1 + 3N_3 \,.$$

By assuming a solution of the form $\mathbf{N} = \mathbf{a}e^{\lambda t}$, where

$$\mathbf{N} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} ,$$

deduce that there is a matrix A such that

$$A\mathbf{a} = \lambda\mathbf{a} \,.$$

[5 marks]

Show that two of the eigenvalues of A are -1 and 3, and find the other eigenvalue and all the eigenvectors.

[15 marks]

Thence write down the general solution of the equations for $\mathbf{N}$.

[4 marks]

If the initial condition at time $t = 0$ is that $\mathbf{N} = (N_0, 0, 0)$ show that, when $t \gg 0$,

$$N_1 = N_2 = N_3 = \frac{N_0}{3}e^{5t} \,.$$

[6 marks]

4) Calculate div$\mathbf{A}$ and curl $\mathbf{A}$ when $\mathbf{A} = x\mathbf{j} - z\mathbf{k}$.

[5 marks]

The transformation from Cartesian coordinates (x, y, z) to spherical polar coordinates (r, θ, ϕ) is given by

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta\,.$$

Show that the Jacobian of the transformation is $r^2\sin\theta$.

[6 marks]

Stokes' theorem states that

$$\int_S \mathrm{curl}\,\mathbf{A}.d\mathbf{S} = \int_C \mathbf{A}.d\mathbf{r}\ ,$$

where $\mathbf{A}$ is a vector field and C is the boundary of a regular open surface S. Verify Stokes' theorem directly for the given vector field $\mathbf{A}$ when S is the surface of the upper half of the sphere of radius $r = R$ and C is the circle in the (x, y)-plane of radius R.

[13 marks]

Use Gauss' theorem to show that

$$\int_{S'} \mathbf{A}.d\mathbf{S} = -2\pi R^3/3\,,$$

where S' is the closed surface given by S (above) and the (x, y)-plane.

[6 marks]

5) The Fourier cosine series of an even function $f(x)$ in the range $-T/2 \leq x \leq T/2$ has the form

$$F(f(x)) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n \cos\left(\frac{2n\pi x}{T}\right)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2n\pi x/T)dx\,,$$

for $n = 0, 1, 2 \ldots$.
Show that the series for the function

$$f(x) = \begin{cases} -x, & -T/2 < x < 0, \\ x, & 0 < x < T/2 \end{cases}$$

is

$$F(f(x)) = \frac{T}{4} + \frac{T}{\pi^2}\sum_{n\geq 1}\frac{((-1)^n - 1)}{n^2}\cos(2n\pi x/T)\,.$$

[16 marks]

Sketch the Fourier series representation of $f(x)$ in the interval $-\frac{3}{2}T \leq x \leq \frac{3}{2}T$. Add to your sketch the function obtained by including only the first two terms of the Fourier series.

[7 marks]

By considering the value of the Fourier series at $x = 0$, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}\,.$$

[7 marks]