King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2250 Mathematical Methods in Physics I

January 2006

Time allowed: THREE Hours

Candidates should answer ALL parts of SECTION A, and no more than TWO questions from SECTION B. No credit will be given for answering further questions.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

The following information defines terms used in this examination and may be of use.

- The Jacobian of the transformation from variables (x, y) to (u, v) is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

- The flux integral

$$J = \int\int_S (\mathbf{F} \cdot \mathbf{n})\, \mathrm{d}S$$

over the surface S specified parametrically via identities $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$, is calculated using

$$J = \int\int (F_x J_x + F_y J_y + F_z J_z)\, \mathrm{d}u\mathrm{d}v$$

applying appropriate limits for parameters u and v, where

$$J_x = \frac{\partial(y,z)}{\partial(u,v)},\ J_y = \frac{\partial(z,x)}{\partial(u,v)},\ J_z = \frac{\partial(x,y)}{\partial(u,v)}$$

are the corresponding Jacobians.

- The *polar coordinates* (r, θ) on the $(x, y)-$ plane are defined by the transformation equations

$$x = r\cos\theta,\ \ y = r\sin\theta$$

SECTION A – Answer ALL parts of this section

1.1) Given the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

find the similarity transformation that diagonalises A. Write down the resulting diagonal matrix.

[7 marks]

1.2) State what is meant by a conservative and solenoidal vector fields and verify that the vector field $\mathbf{F}_1 = \left(y^2 z, z^2 x, x^2 y\right)$ is solenoidal, while $\mathbf{F}_2 = (\sin z, 2y, x\cos z)$ is conservative.

[6 marks]

1.3) A robot standing on a hill whose height $H(x, y) = \exp\left(-x^2 - y^2\right)$, was taught to make jumps of fixed length of λ =0.1 along the direction of steepest downward slope of $H(x, y)$. Assuming that initially the robot was at the point $(1, 1)$, determine its position after 2 steps.

[7 marks]

1.4) Calculate the integral

$$\int\int \frac{e^{-\alpha\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \mathrm{d}x\mathrm{d}y$$

over the entire x, y-plane using an appropriate change of coordinates.

[6 marks]

1.5) What does the statement, that the function $f(x, y)$ is a homogeneous function of degree n, mean? Using an appropriate substitution, integrate the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x^2 - y^2}$$

to find an equation relating the variables x, y up to a constant.

[7 marks]

1.6) Calculate the flux of the vector field $\mathbf{F} = (x, y, 0)$ through the closed surface bounded by the cylinder $x^2 + y^2 = R^2$ of radius R and the two planes $z = 1$ and $z = -1$.

[7 marks]

SECTION B – Answer TWO questions

2) Consider a gas of molecules A with an initial concentration n_0. Upon heating, each molecule A breaks down into a stable molecule C and a metastable species B, that in turn dissociates into a molecule C and an unknown fragment that is not of interest. The concentrations $A(t)$, $B(t)$ and $C(t)$ of the three species satisfy the following equations:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -kA$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{k}{2}A - \frac{k}{2}B$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{k}{2}A + \frac{k}{2}B$$

a) Write the equations in a matrix form $\frac{\mathrm{d}N}{\mathrm{d}t} = UN$ where N is a vector and U a matrix.

[2 marks]

b) Assuming an exponential solution, $N(t) = Ye^{\lambda t}$, show that λ and Y can be obtained by solving an eigenproblem for the matrix U.

[2 marks]

c) Show that the eigenvalues λ_i $(i = 1, 2, 3)$ of the matrix U are $\lambda_1 = 0$, $\lambda_2 = -k$, $\lambda_3 = -k/2$, while the corresponding eigenvectors Y_i can be chosen as

$$Y_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; Y_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; Y_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

[11 marks]

d) Construct all elementary solutions of the equation and thus write down its general solution.

[5 marks]

e) Calculate the particular solutions for the concentrations that correspond to the initial conditions.

[6 marks]

f) Sketch the solutions you calculated, explaining in words the behaviour of each concentration, and finding the time at which the concentration of B is maximal.

[4 marks]

3)

Consider the following differential equation (DE):

$$y'' + y' - 2y = xe^x + \sin x$$

a) Describe the type of this DE.

[2 marks]

b) Show that its complementary solution is

$$y = C_1 e^x + C_2 e^{-2x},$$

where C_1 and C_2 are arbitrary constants.

[4 marks]

c) Determine the particular integral solution.

[16 marks]

d) Hence, state the general solution.

[2 marks]

e) Obtain the particular solution which satisfies the initial conditions $y(0) = 0$ and $y'(0) = 0$.

[6 marks]

4) Given the vector force field $\mathbf{F} = (F_x, F_y, F_z) = (2xz, 2yz, x^2 + y^2)$,

a) Show that $\mathbf{F}$ is conservative.

[2 marks]

b) State the value of the line integral $\oint_L \mathbf{F} \cdot \mathrm{d}\mathbf{l}$ over any closed path L.

[2 marks]

c) Explicitly calculate the line integral along the closed path $x^2 + z^2 = 1$ and $y = 0$ using polar coordinates.

[6 marks]

d) Prove that if the line integral along any closed path is zero, then the line integral between points A and B does not depend on the particular path A→B chosen.

[6 marks]

e) Verify that $\mathrm{d}U = F_x \mathrm{d}x + F_y \mathrm{d}y + F_z \mathrm{d}z$ is the exact differential.

[7 marks]

f) Find, up to a constant, the function $U(x, y, z)$ giving rise to the exact differential above.

[5 marks]

g) State the relationship between the field $\mathbf{F}$ and the function $U(x, y, z)$; what is the latter function U called?

[2 marks]