King's College London

UNIVERSITY OF LONDON

This paper is part of an examination of the College counting towards the award of a degree. Examinations are governed by the College Regulations under the authority of the Academic Board.

BSc EXAMINATION

CP/2250 Mathematical Methods in Physics I

JANUARY 2002

Time allowed: **THREE** HOURS

Candidates must answer any SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each question or part of a question is indicated in square brackets.

Separate answer books **must** be used for each section of the paper.

You must **not** use your own calculator for this paper. Where necessary a College calculator will be supplied.

TURN OVER WHEN INSTRUCTED

SECTION A — answer any SIX parts of this section

1.1 By using an integrating factor or otherwise, find the solution of the differential equation

$$\frac{dy}{dx} + xy = x\exp(-x^2/2)\,,$$

which satisfies the boundary condition that $y = 1$ when $x = 0$.

[7 marks]

1.2 Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0\,,$$

which satisfies the boundary conditions that $y = 1$ and $dy/dx = 0$ when $x = 0$.

[7 marks]

1.3 Given the scalar field

$$\phi = \frac{1}{1 + x^2 + y^4}\,,$$

find the directional derivative of ϕ at the point $(1, 1)$ in the y-direction.

[7 marks]

1.4 Is the vector field $\mathbf{E} = z\cos x\,\mathbf{i} + x\sin y\,\mathbf{j} + xy\,\mathbf{k}$ irrotational or solenoidal or neither?

[7 marks]

1.5 Calculate the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}.$$

[7 marks]

1.6 By transforming to plane polar coordinates evaluate the integral

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}dydx\,.$$

[7 marks]

1.7 Given the vector field $\mathbf{E} = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$ calculate the line integral $\int_C \mathbf{E}.d\mathbf{r}$ where C is the straight line from (0,0,0) to (1,1,0).

[7 marks]

1.8 The Fourier series representation of the function

$$f(x) = \begin{cases} 1+x, & -1 \leq x \leq 0 \\ 1-x, & 0 \leq x \leq 1 \end{cases}$$

is

$$F(f(x)) = \frac{1}{2} + \frac{2}{\pi^2} \sum_{n \geq 1} \frac{((-1)^{n+1} + 1)}{n^2} \cos n\pi x \,.$$

Sketch the function $F(f(x))$ in the interval $-3 \leq x \leq +3$. By considering the value of F at $x = 0$, find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots.$$

[7 marks]

SECTION B – answer TWO questions

2. The behaviour of a certain forced damped simple harmonic oscillator is determined by the differential equation

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + 4y = 4\cos 2t\,,$$

where $0 < k \ll 1$ is the damping constant and y is the amplitude of the motion. Find the general solution of this equation for the amplitude as a function of time t.

[8 marks]

What is the solution applicable when $kt \gg 1$?

[2 marks]

If the damping term is absent, the differential equation for the motion becomes

$$\frac{d^2y}{dt^2} + 4y = 4\cos 2t\,.$$

Use the D-operator method to deduce that a particular integral of the equation is

$$y_I(t) = \frac{1}{4}\cos 2t + t\sin 2t,$$

and determine the general solution of the equation.

[12 marks]

What is the dominant term of the solution as $t \to \infty$? Why does the absence of damping change the behaviour so radically?

[8 marks]

3. If $\mathbf{x}$ is an eigenvector of the matrix A with eigenvalue λ, prove that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ for all $n \geq 1$.

[5 marks]

A certain system can exist in two possible states. The vector $\mathbf{y}_0 = \begin{pmatrix} p \\ 1-p \end{pmatrix}$, with $0 \leq p \leq 1$, represents the probabilities of the system being in one or other of the two states at time $t = 0$. At each time step the system moves to a new state determined by a transition matrix A; that is, at time $t = 1$, the system is in state $\mathbf{y}_1$ where $\mathbf{y}_1 = A\mathbf{y}_0$. The transition matrix is

$$A = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix} .$$

Given that one eigenvalue of A is 1, find the other eigenvalue and the corresponding unnormalised eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

[15 marks]

The initial state $\mathbf{y}_0$ can be expressed as a linear combination of the eigenvectors that is, $\mathbf{y}_0 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. Find the coefficients a_1 and a_2.

[5 marks]

Hence deduce that, for a very large number n of time steps,

$$A^n\mathbf{y}_0 \to \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} .$$

[5 marks]

4. Calculate div$\mathbf{A}$ and curl$\,\mathbf{A}$ when $\mathbf{A} = yz\mathbf{i} + xz\mathbf{j} + z\mathbf{k}$.

[4 marks]

Gauss's theorem states that

$$\int_V \text{div}\,\mathbf{A} dv = \int_S \mathbf{A}.d\mathbf{S} ,$$

where $\mathbf{A}$ is a vector field and V is the volume enclosed by a regular closed surface S. Verify Gauss's theorem directly for the vector field $\mathbf{A}$ given above, when V is the volume of a cube with one corner at the point (0,0,0) and three other corners at (1,0,0), (0,1,0) and (0,0,1).

[11 marks]

State Stokes' theorem.

[4 marks]

For the same vector field $\mathbf{A}$, verify Stokes' theorem by evaluating a surface integral and a line integral, where the surface is the square in the $y = 1$ plane whose corners are at the points (0,1,0), (1,1,0), (0,1,1) and (1,1,1).

[11 marks]

5. The Fourier series of a function $f(x)$ in the range $-T/2 \leq x \leq T/2$ has the form

$$F(f(x)) = a_0/2 + \sum_{n\geq 1}(a_n \cos(2n\pi x/T) + b_n \sin(2n\pi x/T))$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2n\pi x/T)dx\,,$$
$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\sin(2n\pi x/T)dx\,.$$

Show that the Fourier series of the function $f(x) = L^2 - x^2$ when $-L \leq x \leq L$ is

$$F(f(x)) = \frac{2}{3}L^2 + \frac{4L^2}{\pi^2}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x/L)\,.$$

[15 marks]

By choosing suitable values for x, find the sums of the series

$$S_1 = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$
$$S_2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\,.$$

[10 marks]

Hence deduce that

$$\frac{\pi^2}{8} = S_3 = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\,.$$

[5 marks]