King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2250 Mathematical Methods in Physics I

Summer 2001

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Find the solution of the differential equation

$$2x\frac{dy}{dx} - y = \frac{1}{y}\,,$$

which satisfies the boundary condition that $y = 0$ when $x = 1$.

[7 marks]

1.2) Find the solution of the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0\,,$$

which satisfies the boundary conditions that $y = 0$ and $dy/dt = 2$ when $t = 0$.

[7 marks]

1.3) Given that the scalar field $\phi = 1/u$ where $u = (x^3 + y^3)^{1/3}$, calculate grad ϕ.

[7 marks]

1.4) Find div $\mathbf{E}$ and curl $\mathbf{E}$ when $\mathbf{E} = x\sin y\mathbf{i} + \cos y\mathbf{j} + xy\mathbf{k}$. Is the field $\mathbf{E}$ irrotational or solenoidal or neither?

[7 marks]

1.5) Calculate the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}.$$

[7 marks]

1.6) A circular laminar of radius R centred at the origin has a mass density $\rho = \rho_0\sqrt{x^2 + y^2}$, where ρ_0 is a constant. Using plane polar coordinates calculate the mass of the laminar.

[7 marks]

1.7) Given the vector field $\mathbf{E} = \mathbf{i} - z\mathbf{j} - y\mathbf{k}$ calculate the surface integral $\int_S \mathbf{E}.d\mathbf{S}$ where S is the square in the (x, y)-plane with sides of unit length with one corner at the origin and the opposite corner at the point (1,1,0).

[7 marks]

1.8) The Fourier series representation of the function $f(x) = |x|$ when $-T/2 < x < T/2$ is

$$F(f(x)) = \frac{T}{4} + \frac{T}{\pi^2} \sum_{n \geq 1} \frac{((-1)^n - 1)}{n^2} \cos 2n\pi x/T\,.$$

Sketch the function $F(f(x))$ when $-3T/2 < x < 3T/2$ and find the sum of the series

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\,.$$

[7 marks]

SECTION B – Answer TWO questions

2) The behaviour of a damped simple harmonic oscillator acted on by an external force $F(t)$ is determined by the differential equation

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + y = F(t)\,,$$

where k is a damping constant.. What is the general solution of this equation when $k < 1$ and $F(t) = 0$?

[6 marks]

What is the solution which satisfies the boundary conditions that $y = 1$ and $dy/dt = 0$ when $t = 0$?

[7 marks]

Briefly describe in words the behaviour of your solution for y as a function of time t.

[2 marks]

Show that the solution has successive maxima and minima at times t_m given by

$$t_m = \frac{n\pi}{\sqrt{1-k^2}}\,,$$

where $n = 0, 1, 2, \ldots$.

[6 marks]

If the driving force is given by $F(t) = \sin\omega t$, where ω is an angular frequency, find the solution for $y(t)$ applicable when $kt \gg 1$.

[9 marks]

3) Three bacterial species eat each other but are also supplied with food from an external source. The rate of change of the number N_i of each species i is given by the set of coupled differential equations

$$\frac{dN_1}{dt} = -N_1 + 2N_2 + N_3$$
$$\frac{dN_2}{dt} = 2N_1 + 3N_2$$
$$\frac{dN_3}{dt} = N_1 + 3N_3\,.$$

By assuming a solution of the form $\mathbf{N} = \mathbf{a}e^{\lambda t}$, where

$$\mathbf{N} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix},$$

deduce that there is a matrix A such that

$$A\mathbf{a} = \lambda\mathbf{a}\,.$$

[5 marks]

Show that two of the eigenvalues of A are -2 and 3, and find the other eigenvalue and all the eigenvectors.

[15 marks]

Thence write down the general solution of the equations for $\mathbf{N}$.

[4 marks]

If the initial condition at time $t = 0$ is that $\mathbf{N} = (N_0, 0, 0)$ show that, when $t \gg 0$,

$$2N_1 = N_2 = 2N_3 = \frac{N_0}{3}e^{4t}\,.$$

[6 marks]

4) Calculate div$\mathbf{A}$ and curl $\mathbf{A}$ when $\mathbf{A} = -y\mathbf{i} + z\mathbf{k}$, and verify directly that

$$\text{curl}\,\text{curl}\,\mathbf{A} = \text{grad}\,\text{div}\,\mathbf{A} - \nabla^2\mathbf{A}.$$

[7 marks]

The transformation from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z') is given by

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z'\,.$$

Find the Jacobian of the transformation.

[4 marks]

Stokes' theorem states that

$$\int_S \text{curl}\,\mathbf{A}.d\mathbf{S} = \int_C \mathbf{A}.d\mathbf{r}\ ,$$

where $\mathbf{A}$ is a vector field and C is the boundary of a regular open surface S. Verify Stokes' theorem directly for the vector field $\mathbf{A}$ given above, when S is the surface of the tin can bounded by $x^2 + y^2 = 1$, $z = 0$ and $z = a$, where a is a constant, with the open end of the can at $z = 0$.

[13 marks]

Use Gauss' theorem to show that

$$\int_{S'} \mathbf{A}.d\mathbf{S} = \pi a,$$

where S' is the closed surface given by S (above) and the plane $z = 0$.

[6 marks]

5) The complex Fourier series of a function $f(x)$ in the range $0 < x < T$ has the form

$$F(f(x)) = \sum_{n=-\infty}^{\infty} c_n \exp(2in\pi x/T)$$

where

$$c_n = \frac{1}{T}\int_0^T f(x)\exp(-2in\pi x/T)dx\,.$$

Sketch the function

$$f(x) = x$$

for x in the interval $0 < x < T$, and sketch the Fourier series representation of $f(x)$ over the range $-T < x < 3T$.

[5 marks]

Show that the Fourier series representation of $f(x)$ is given by

$$F(f(x)) = \frac{T}{2} - \frac{T}{2i\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\exp(2in\pi x/T)}{n}$$
$$= \frac{T}{2} - \frac{T}{\pi}\sum_{n\geq 1} \frac{\sin(2n\pi x/T)}{n}\,.$$

[10 marks]

Does the value of the Fourier series representation agree with what you expect at the points $x = T/2$ and $x = T$?

[5 marks]

By considering the value of the Fourier series at $x = T/4$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots\,.$$

[5 marks]

Use Parseval's theorem to show that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}\,.$$

[5 marks]