King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2250 Mathematical Methods in Physics I

Summer 2000

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) By separating variables, find the solution of the differential equation

$$3y^2\frac{dy}{dx} - 2xy^3 = 2x\,,$$

which satisfies the boundary condition that $y = 0$ when $x = 0$.

[7 marks]

1.2) Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0\,,$$

which satisfies the boundary conditions that $y = 0$ and $dy/dx = 4$ when $x = 0$.

[7 marks]

1.3) Given the scalar field

$$\phi = \frac{1}{1 + x^2 + y^2}\,,$$

find the directional derivative of ϕ at the point $(1, 1)$ in the x-direction.

[7 marks]

1.4) Show that the vector field $\mathbf{E} = zx\cosh y\mathbf{i} - z\sinh y\mathbf{j} + xy\mathbf{k}$ is solenoidal.

[7 marks]

1.5) Calculate the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\,.$$

[7 marks]

1.6) By transforming to plane polar coordinates evaluate the integral

$$\int_{y=0}^{1}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\,dxdy\,.$$

[7 marks]

1.7) Given the vector field $\mathbf{E} = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$ calculate the line integral $\int_C \mathbf{E}.d\mathbf{r}$ where C is the straight line from (0,0,0) to (1,1,1).

[7 marks]

1.8) The Fourier series representation of the function

$$f(x) = x^2, \quad -T/2 \leq x \leq T/2\,,$$

is

$$F(f(x)) = \frac{T^2}{12} + \frac{T^2}{\pi^2} \sum_{n \geq 1} \frac{(-1)^n}{n^2} \cos 2n\pi x/T\,.$$

Sketch the function $F(f(x))$ in the interval $-3T/2 \leq x \leq 3T/2$. By considering the value of F at $x = T/2$, find the sum of the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\,.$$

[7 marks]

SECTION B – Answer TWO questions

2) The behaviour of a forced damped simple harmonic oscillator is determined by the differential equation

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + y = 2\cos t\,,$$

where $k \ll 1$ represents a damping term and y is the amplitude of the motion. Find the general solution of this equation for the amplitude as a function of time t.

[8 marks]

What is the solution applicable when $kt \gg 1$?

[2 marks]

If the damping term is absent, the differential equation for the motion becomes

$$\frac{d^2y}{dt^2} + y = 2\cos t\,.$$

Show that a particular integral of the equation is

$$y_I(t) = -\frac{1}{2}\cos t + t\sin t,$$

and determine the general solution of the equation.

[12 marks]

What is the dominant term of the solution as $t \to \infty$? Why does the absence of damping change the behaviour so radically?

[8 marks]

3) If $\mathbf{x}$ is an eigenvector of the matrix A with eigenvalue λ, prove that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ for all $n \geq 1$.

[5 marks]

A system can exist in two possible states. The vector $\mathbf{y}_0 = \begin{pmatrix} p \\ 1-p \end{pmatrix}$, with $0 \leq p \leq 1$, represents the probabilities of the system being in one or other of the two states at time $t = 0$. At each time step the system moves to a new state determined by a transition matrix A, that is, at time $t = 1$ the system is in state $\mathbf{y}_1$ where $\mathbf{y}_1 = A\mathbf{y}_0$. The transition matrix is

$$A = \begin{pmatrix} 2/3 & 1/2 \\ 1/3 & 1/2 \end{pmatrix} .$$

Given that one eigenvalue of A is 1, find the other eigenvalue and the corresponding unnormalised eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

[15 marks]

The initial state $\mathbf{y}_0$ can be expressed as a linear combination of the eigenvectors, that is, $\mathbf{y}_0 = a_1\mathbf{x}_1 + a_2\mathbf{x}_2$. Find the coefficients a_1 and a_2.

[5 marks]

Hence deduce that, for a very large number n of time steps,

$$A^n\mathbf{y}_0 \to \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} \quad \text{as } n \to \infty .$$

[5 marks]

4) Calculate div$\mathbf{A}$ and curl $\mathbf{A}$ when $\mathbf{A} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$.

[5 marks]

Gauss's theorem states that

$$\int_V \text{div}\, \mathbf{A} dv = \int_S \mathbf{A}.d\mathbf{S} ,$$

where $\mathbf{A}$ is a vector field and V is the volume enclosed by a regular closed surface S. Verify Gauss's theorem directly for the given vector field $\mathbf{A}$ when V is the volume of a cube with one corner at the point (0,0,0) and three other corners at (a,0,0), (0,a,0) and (0,0,a). [Hint: make use of the symmetry of the problem.]

[13 marks]

State Stoke's theorem.

[4 marks]

For the same vector field $\mathbf{A}$, verify Stokes' theorem by evaluating a surface integral and a line integral, where the surface is the square in the $z = 0$ plane whose corners are at the points $(0, 0, 0)$, $(a, 0, 0)$, $(a, a, 0)$ and $(0, a, 0)$.

[8 marks]

5) The complex Fourier series of a function $f(x)$ in the range $-T/2 \leq x \leq T/2$ has the form

$$F(f(x)) = \sum_{n=-\infty}^{\infty} c_n e^{2in\pi x/T}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} f(x) e^{-2in\pi x/T} dx\,.$$

Show that the complex Fourier series for the function

$$f(x) = \begin{cases} -A, & -T/2 < x < 0, \\ A, & 0 < x < T/2, \end{cases}$$

is

$$F(f(x)) = \frac{A}{i\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(1-(-1)^n)}{n} e^{2in\pi x/T}.$$

[16 marks]

Use Parseval's theorem to show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\,.$$

[14 marks]