King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2210 MATHEMATICAL METHODS IN PHYSICS II

Summer 1998

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) A two-dimensional curvilinear coordinate system (q_1, q_2) is defined by the transformation equations

$$x = 2q_1 q_2\,,$$
$$y = q_1^2 - q_2^2\,,$$

where $0 \leq q_1 \leq \infty$ and $-\infty < q_2 < \infty$. Determine the unit base vectors for this system.

[7 marks]

1.2) Show that, for a general curvilinear orthogonal coordinate system (q_1, q_2, q_3), the gradient of a scalar field $\psi(q_1, q_2, q_3)$ is given by

$$\operatorname{grad}\psi = \sum_{i=1}^{3} \frac{\mathbf{e}_i}{h_i}\frac{\partial\psi}{\partial q_i}\,,$$

where $\{\mathbf{e}_i\,; i = 1, 2, 3\}$ and $\{h_i\,; i = 1, 2, 3\}$ denote the sets of unit base vectors and scale factors respectively for the coordinate system.

[7 marks]

1.3) State the general *filtering theorem* for the Dirac delta function $\delta(x)$. Hence evaluate the integral

$$\int_{-\infty}^{\infty} \delta(4t + \pi)\sin(2t)\,dt\,.$$

[7 marks]

1.4) Define the *Fourier transform* $\mathcal{F}[f(t)]$ of a function $f(t)$ which is defined on the interval $-\infty < t < \infty$. Calculate the Fourier transform of the function

$$f(t) = H(t)\exp(-4\pi t)\,,$$

where

$$H(t) = 0\,, \quad t < 0$$
$$= 1\,, \quad t \geq 0$$

is the Heaviside step function.

[7 marks]

1.5) Define the Laplace transform $\mathcal{L}[f(t)] = F(p)$ of a function $f(t)$ which is defined on the interval $0 \leq t < \infty$. Determine the inverse $f(t)$ of the Laplace transform

$$F(p) = \frac{p}{(p-1)(p+4)} .$$

[7 marks]

[It may be assumed that $\mathcal{L}[\exp(at)] = 1/(p-a)$, where a is a constant and $p > a$.]

1.6) Explain what is meant by a *regular singular point* of a linear differential equation of second order. Classify all the singular points of the differential equation

$$x^2(x+4)\,\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + (x-1)\,\frac{\mathrm{d}y}{\mathrm{d}x} + x(x-2)\,y = 0 .$$

[7 marks]

1.7) Prove that if $\psi = f(x-ct)$, where c is a constant and $f(w)$ is an **arbitrary** function of w , then

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 \psi}{\partial t^2} .$$

[7 marks]

1.8) Use the generating function for Legendre polynomials

$$(1 - 2\mu t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu) t^n ,$$

where $-1 \leq \mu \leq 1$ and $|t| \leq 1$, to prove that

$$P_n(-\mu) = (-1)^n\, P_n(\mu)$$

for all $n = 0, 1, 2, \ldots$.

[7 marks]

SECTION B – Answer TWO questions

2) Show that the Fourier transform $\mathcal{F}[f(t)] = F(\nu)$ of an **even** function $f(t)$ can be written in the form

$$\mathcal{F}[f(t)] = 2\int_0^\infty f(t)\,\cos(2\pi\nu t)\,\mathrm{d}t\,.$$

[7 marks]

Prove that the Fourier transform $F(\nu)$ of the function

$$\begin{aligned} f(t) =& 1 - |t| \quad \text{for} \quad 0 \le |t| < 1 \\ =& 0\,, \qquad\quad \text{otherwise} \end{aligned}$$

is given by

$$F(\nu) = \left[\frac{\sin(\pi\nu)}{\pi\nu}\right]^2 .$$

[14 marks]

Use the inverse Fourier transform to evaluate the integral

$$\int_0^\infty \cos x \left(\frac{\sin x}{x}\right)^2 \mathrm{d}x\,.$$

[9 marks]

3) Prove that the Laplace transforms of the functions te^{at}, $\sin(at)$ and $\cos(at)$ are given by

$$\begin{aligned} \mathcal{L}[te^{at}] =& \frac{1}{(p-a)^2}\,, \\ \mathcal{L}[\sin(at)] =& \frac{a}{p^2+a^2}\,, \\ \mathcal{L}[\cos(at)] =& \frac{p}{p^2+a^2}\,, \end{aligned}$$

where a is a positive constant and $p > a$.

[9 marks]

Use the Laplace transform method to determine the solution $f(t)$ of the differential equation

$$\frac{d^2f}{dt^2} + 6\frac{df}{dt} + 9f = \sin t\,,$$

which satisfies the initial conditions $f(0) = 0$ and $f'(0) = 0$. Derive any formulae that are needed in the calculation.

[21 marks]

4) Use the method of Frobenius to derive **two** independent series solutions of the differential equation

$$3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x-1)y = 0$$

in powers of x.

[24 marks]

Show that the series solutions converge for all $|x| < \infty$.

[6 marks]

5) Apply the method of separation of variables to the Laplace equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0\,,$$

where $\psi = \psi(r, \theta, \phi)$ and (r, θ, ϕ) are spherical polar coordinates. Hence show that the physically acceptable **product** solutions of the Laplace equation, which are axially symmetric about the z axis and finite at the origin $r = 0$, are given by

$$\psi(r, \theta, \phi) = r^n P_n(\cos\theta)\,,$$

where $n = 0, 1, 2, \ldots$, and $P_n(\mu)$ denotes a Legendre polynomial in the variable $\mu = \cos\theta$.

[It may be assumed that the differential equation

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left(U \sin^2\theta\right)\Theta = 0\,,$$

only has a physically acceptable solution when the separation constant $U = n(n+1)$ with $n = 0, 1, 2, \ldots$, and that this solution is given by the Legendre polynomial $P_n(\mu)$ with $\mu = \cos\theta$.]

[20 marks]

Determine the particular solution $\psi = \psi(r, \theta, \phi)$ of the Laplace equation which is single-valued and finite in the region $0 \le r \le a$, and satisfies the boundary condition

$$\psi(a, \theta, \phi) = \cos^2\theta\,,$$

on the surface of the sphere $r = a$.

[Note that the first three Legendre polynomials are $P_0(\mu) = 1$, $P_1(\mu) = \mu$ and $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$.]

[10 marks]