King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2210 MATHEMATICAL METHODS IN PHYSICS II

Summer 1997

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Define the *unit base vectors* $\{\mathbf{e}_i; i = 1, 2, 3\}$ and the *scale factors* $\{h_i; i = 1, 2, 3\}$ for a general three-dimensional curvilinear coordinate system.

A particle moving in three dimensions has a position vector $\mathbf{r}(t)$. Show that the velocity of the particle can be written in the form

$$\dot{\mathbf{r}} = \sum_{i=1}^{3} h_i \dot{q}_i \, \mathbf{e}_i \, .$$

[7 marks]

1.2) State the general *filtering theorem* for the Dirac delta function. Hence evaluate the integral

$$\int_{-\infty}^{\infty} \delta(t-1)\,(1+4t^2)^{-1}\,dt\,.$$

[7 marks]

1.3) A periodic function $f(t)$ with a fundamental period $T = 2\pi$ can be represented by the complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{int} \, ,$$

where c_n is a constant. Show that this Fourier series can be written in the alternative form

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \, \cos(nt) + b_n \, \sin(nt)] \, .$$

Obtain formulae for the constants $\{a_n; n = 0, 1, 2, \ldots\}$ and $\{b_n; n = 1, 2, \ldots\}$ in terms of c_n and c_{-n}.

[7 marks]

1.4) Define the *Fourier Transform* $\mathcal{F}[f(t)]$ of a function $f(t)$ which is defined on the interval $-\infty < t < \infty$. Calculate the Fourier transform of the Dirac delta function $\delta(2t+1)$.

[7 marks]

1.5) Define the *Laplace transform* $\mathcal{L}[f(t)] = F(p)$ of a function $f(t)$ which is defined on the interval $0 \leq t < \infty$. Determine the inverse $f(t)$ of the Laplace transform

$$F(p) = \frac{1}{(p+1)(p-3)} .$$

[7 marks]

[It may be assumed that $\mathcal{L}[e^{at}] = 1/(p-a)$, where a is a constant and $p > a$.]

1.6) Explain what is meant by a *regular singular point* of a linear differential equation of second order. Classify all the singular points of the differential equation

$$x^3(x-1)\,\frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} + x(x+2)\,\frac{\mathrm{d}y}{\mathrm{d}x} + (x-2)y = 0 .$$

[7 marks]

1.7) Determine the *general* solution $R(r)$ of the differential equation

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 ,$$

by using the trial solution $R(r) = r^s$, where $n = 0, 1, 2, \ldots$ and s is a constant.
[7 marks]

1.8) Use the generating function for Legendre polynomials

$$(1 - 2\mu t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(\mu) t^n ,$$

where $-1 \leq \mu \leq 1$ and $|t| \leq 1$, to obtain formulae for $P_0(\mu)$, $P_1(\mu)$ and $P_2(\mu)$.
[7 marks]

SECTION B – Answer TWO questions

2) Show that, for a general curvilinear orthogonal coordinate system (q_1, q_2, q_3), the gradient of a scalar field $\psi(q_1, q_2, q_3)$ can be written as

$$\operatorname{grad}\psi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\psi}{\partial q_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\psi}{\partial q_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\psi}{\partial q_3},$$

where $\{h_i; i = 1,2,3\}$ and $\{\mathbf{e}_i; i = 1,2,3\}$ denote the sets of scale factors and unit base vectors respectively for the coordinate system.

[10 marks]

A particular curvilinear orthogonal coordinate system (q_1, q_2, q_3) is defined by the transformation equations

$$x = q_1 q_2 \cos q_3\,,$$
$$y = q_1 q_2 \sin q_3\,,$$
$$z = \tfrac{1}{2}(q_1^2 - q_2^2)\,,$$

where $q_1 \geq 0$, $q_2 \geq 0$ and $0 \leq q_3 < 2\pi$. Determine the scale factors $\{h_i; i = 1,2,3\}$ and unit base vectors $\{\mathbf{e}_i; i = 1,2,3\}$ for this system.

[12 marks]

Hence calculate the gradient of the scalar field

$$\psi(q_1, q_2, q_3) = (q_1^2 + q_2^2)\cos q_3$$

at the point P which has curvilinear coordinates $q_1 = 1$, $q_2 = 1$ and $q_3 = \frac{\pi}{4}$. Express your answer in terms of the Cartesian unit vectors **i**, **j** and **k**.

[8 marks]

3) Use the method of Frobenius to derive **two** independent series solutions of the differential equation

$$2x^2\frac{d^2y}{dx^2} + x(2x+3)\frac{dy}{dx} + (3x-1)y = 0$$

in powers of x.

[24 marks]

Use the ratio test to prove that the general series solution converges for all $|x| < \infty$.

[6 marks]

4) Show that the Fourier transform $\mathcal{F}[f(t)]$ of an **even** function $f(t)$ can be written in the form

$$\mathcal{F}[f(t)] = 2\int_0^\infty f(t)\cos(2\pi\nu t)\,dt\,.$$

[6 marks]

Determine the Fourier transform of the even function $f(t)$ defined by

$$\begin{aligned} f(t) &= 1 \quad \text{for} \quad 0 \le |t| \le 1\,, \\ &= -1 \quad \text{for} \quad 1 < |t| \le 2\,, \\ &= 0 \quad \text{for} \quad 2 < |t| < \infty\,. \end{aligned}$$

[14 marks]

Use the inverse Fourier transform to prove that

$$\int_0^\infty \frac{\sin x}{x}(1-\cos x)\,dx = \frac{\pi}{4}\,.$$

[10 marks]

5) Prove that the Laplace transforms of the functions e^{at} and te^{at}, where a is a constant, are

$$\mathcal{L}[e^{at}] = \frac{1}{p-a}\,,$$

and

$$\mathcal{L}[te^{at}] = \frac{1}{(p-a)^2}\,,$$

provided that $p > a$.

[8 marks]

Use the Laplace transform method to determine the solution $f(t)$ of the differential equation

$$\frac{d^2 f}{dt^2} - 3\frac{df}{dt} + 2f = 4\,e^{2t}\,,$$

which satisfies the initial conditions $f(0) = -3$ and $f'(0) = 5$. Derive any formulae that are needed in the calculation.

[22 marks]