King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 1998

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A,
and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Values of physical constants

Planck constant	h	$= 6.626 \times 10^{-34}\,\mathrm{J\,s}$
speed of light	c	$= 2.998 \times 10^{8}\,\mathrm{m\,s^{-1}}$

SECTION A – Answer SIX parts of this section

1.1) The radiation emitted by a He-Ne laser has wavelength $\lambda = 633\,\mathrm{nm}$. How many photons are emitted per second by a laser with a power of $0.5\,\mathrm{mW}$?

[7 marks]

1.2) Explain what is meant by a *complete orthonormal set* of functions. What is an *eigenfunction expansion*?

[7 marks]

1.3) Explain what is meant by the *correspondence principle* and by the *complementarity principle.*

[7 marks]

1.4) The possible energies of a particle in a box with sides $(2a, 2a, a)$ are given by

$$E_{n_1,n_2,n_3} = ({n_1}^2 + {n_2}^2 + 4{n_3}^2)E\,,$$

where n_1, n_2, n_3 are positive integers and E is a constant. Find the energy ϵ_0 of the ground state in terms of the energy E and show that the energies of the two lowest *non-degenerate* excited levels are $\epsilon_1 = 12E$ and $\epsilon_2 = 18E$. How many degenerate levels lie between ϵ_0 and ϵ_2, and what are their degeneracies?

[7 marks]

1.5) At a given instant, a quantum harmonic oscillator is in a state described by the normalized wave function

$$\psi(x) = \sqrt{\frac{2}{3}}u_0(x) + \frac{1}{2}\sqrt{\frac{1}{3}}u_2(x) + \frac{1}{2}u_3(x)\;,$$

where $u_n(x)$ is the normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $\frac{17}{12}\hbar\omega$.

[7 marks]

1.6) Starting from the definition of the orbital angular momentum operator

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}\,,$$

derive expressions for the Cartesian components $\boldsymbol{L}_x, \boldsymbol{L}_y$ and $\boldsymbol{L}_z$ in the Schrödinger representation. The operators representing any two components of orbital angular momentum are *incompatible*. What does this mean?

[7 marks]

1.7) An electron is in the unnormalised spin state

$$\psi = \begin{pmatrix} 3 \\ -4i \end{pmatrix}.$$

Normalise ψ and find the expectation value in this state of the spin component

$$\boldsymbol{S}_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is the probability that a measurement of $\boldsymbol{S}_x$ gives the value $+\frac{1}{2}\hbar$?

[7 marks]

1.8) The energy levels of the hydrogen atom are given by

$$E_n = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_o a_o}\frac{1}{n^2}\,, \qquad (n = 1, 2, 3, \ldots)$$

Use the Bohr frequency condition to show that the wavelength of the radiation corresponding to a transition between levels $n = 2$ and $n = 1$ is given by

$$\frac{1}{\lambda_{2,1}} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right).$$

Derive an expression for the constant R. What are the degeneracies of the two levels?

[7 marks]

SECTION B – Answer TWO questions

2) Define an Hermitian operator.

[4 marks]

An Hermitian operator $\mathbf{A}$ corresponding to an observable $\mathcal{A}$ has normalized eigenfunctions u_1 and u_2 with corresponding eigenvalues a_1 and a_2. Similarly, $\mathbf{B}$ corresponding to observable $\mathcal{B}$, has normalized eigenfunctions v_1 and v_2 with corresponding eigenvalues b_1 and b_2. The eigenfunctions are related as follows

$$u_1 = (v_1 + 2v_2)/\sqrt{5}, \qquad u_2 = (2v_1 - v_2)/\sqrt{5}\,.$$

Suppose that when $\mathcal{A}$ is measured, the value a_1 is obtained. If $\mathcal{B}$ is now measured followed by $\mathcal{A}$, show that the probability of obtaining a_1 again is $\frac{17}{25}$.

[26 marks]

3) The *raising* and *lowering* operators $\boldsymbol{L}_+$ and $\boldsymbol{L}_-$ are defined by

$$\boldsymbol{L}_\pm = \boldsymbol{L}_x \pm i\boldsymbol{L}_y\ ,$$

where $(\boldsymbol{L}_x, \boldsymbol{L}_y, \boldsymbol{L}_z)$ are the components of the Hermitian operator $\boldsymbol{L}$ representing the orbital angular momentum. The function $Y_{\ell,m}(\theta,\phi)$ is the normalized eigenfunction of $\boldsymbol{L}_z$ and $\boldsymbol{L}^2$ with eigenvalues $m\hbar$ and $\ell(\ell+1)\hbar^2$, respectively, where $\ell = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots \pm \ell$.

(i) Show that $\boldsymbol{L}_+$ is *not* Hermitian.

[6 marks]

(ii) Show that

$$[\boldsymbol{L}_z, \boldsymbol{L}_+] = \hbar \boldsymbol{L}_+ \qquad \text{and} \qquad [\boldsymbol{L}^2, \boldsymbol{L}_+] = 0\,.$$

[6 marks]

(iii) Using the results in (ii), show that $\boldsymbol{L}_+ Y_{\ell,m}$ is an eigenfunction of $\boldsymbol{L}_z$ and $\boldsymbol{L}^2$ and find the corresponding eigenvalues.

[12 marks]

(iv) By interpreting the results in (iii), justify the name of *raising operator* for $\boldsymbol{L}_+$.

[2 marks]

(v) What properties would you expect the lowering operator $\boldsymbol{L}_-$ to possess?

[4 marks]

You may assume the commutator relations

$$[\boldsymbol{L}_x, \boldsymbol{L}_y] = i\hbar \boldsymbol{L}_z\,, \qquad [\boldsymbol{L}_y, \boldsymbol{L}_z] = i\hbar \boldsymbol{L}_x\,, \qquad [\boldsymbol{L}_z, \boldsymbol{L}_x] = i\hbar \boldsymbol{L}_y$$

and

$$[\boldsymbol{L}^2, \boldsymbol{L}_x] = [\boldsymbol{L}^2, \boldsymbol{L}_y] = [\boldsymbol{L}^2, \boldsymbol{L}_z] = 0\,.$$

4) A beam of particles of mass m and energy E is incident from $x < 0$ upon a potential step at $x = 0$ of height $V_0\ (> E)$. Let

$$k^2 = \frac{2mE}{\hbar^2}\,, \qquad \lambda^2 = \frac{2m}{\hbar^2}(V_0 - E)\,.$$

The incident particles are represented by the wavefunction e^{ikx}. Calculate the reflection coefficient $\mathcal{R}$ and compare your answer with the classical result.

[15 marks]

Show that in the region $x > 0$ the amplitude of the wave function is $2\cos\theta\exp(-\lambda x)$, where $\tan\theta = \lambda/k$.

[12 marks]

What is the net flux of particles in this region?

[3 marks]

5) A quantum particle of mass m moves in one dimension subject to a potential

$$V(x) = \begin{cases} 0\,, & |\,x\,| < a\,, \\ +\infty\,, & |\,x\,| > a\,. \end{cases}$$

The energy eigenvalues are $E_n = \hbar^2\pi^2 n^2/8ma^2$ for $n = 1, 2, 3, \ldots$ and the corresponding orthonormal eigenfunctions are

$$u_n(x) = \begin{cases} a^{-\frac{1}{2}}\cos(n\pi x/2a)\,, & n = 1, 3, 5, \ldots \\ a^{-\frac{1}{2}}\sin(n\pi x/2a)\,, & n = 2, 4, 6, \ldots\,. \end{cases}$$

Suppose that at $t = 0$, the particle is described by the state function

$$\psi(x) = \frac{4}{5}u_1(x) + \frac{3}{5}u_5(x)\,.$$

(i) Verify that $\psi(x)$ is normalized.

[5 marks]

(ii) Write down the state function $\Psi(x, t)$ at time t.

[5 marks]

(iii) Calculate the probabilities of finding the particle at time t with the energies E_n $(n = 1, 2, 3, \ldots)$ and show they are the same as the corresponding probabilities at $t = 0$.

[10 marks]

(iv) Calculate the probability density $|\Psi|^2$ and hence determine how the probability density at the origin varies with t.

[10 marks]