King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 Introduction to Quantum Mechanics

Summer 2003

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Planck constant	$h = 6.626 \times 10^{-34}$ J s
speed of light	$c = 2.998 \times 10^{8}$ m s^{-1}
mass of electron	$m = 9.109 \times 10^{-31}$ kg
charge of electron	$e = -1.602 \times 10^{-19}$ C
Bohr magneton	$\mu_B = 9.273 \times 10^{-24}$ J T^{-1}
g-factor of an electron	$g = 2.0$
mass of a nucleon	$m_p \approx m_n = 1.674 \times 10^{-27}$ kg

The Pauli spin matrices are

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\int_{-\infty}^{\infty} x^{2n} \exp(-\alpha x^2) dx = \frac{\sqrt{\pi}(2n)!}{2^{2n} n! \alpha^{(2n+1)/2}}, \quad n = 1, 2, 3, \ldots.$$

SECTION A – Answer SIX parts of this section

1.1) A particle moves along the x axis in a region of zero potential. Show by substitution in the Schrödinger equation that $u(x) = \exp(ikx)$ is a wavefunction for the particle with energy $E = \hbar^2 k^2/2m$. Given that the momentum operator is $-i\hbar d/dx$, what is the momentum of the particle in terms of k?

[7 marks]

1.2) An electron is trapped in a length $\Delta x = 0.2$ nm. Estimate the uncertainty in its momentum. At time $t = 0$ the barriers trapping the electron are removed. Over what range of x is the electron likely to be found at $t = 1$ ns?

[7 marks]

1.3) A gas that is excited at low-pressure emits photons with a mean energy of 2.4 eV. The typical lifetime of an electron in an excited state of an atom of the gas before photon emission is $\sim 10^{-9}$ s. What is the fractional spread, resulting from the uncertainty principle, in the energy of the emitted photons?

[7 marks]

1.4) The normalised state of a harmonic oscillator in its first excited state is

$$u(x) = Cx \exp\left(-\frac{x^2}{2a^2}\right)$$

where a is a real constant. Show that

$$C = \left(\frac{2}{a^3\pi^{1/2}}\right)^{1/2}.$$

[7 marks]

1.5) What is meant by the statement that two states are orthogonal to each other? Two states of a particle have different energies. Show that the states are orthogonal to each other.

[7 marks]

1.6) At time $t = 0$ a particle has the normalised state

$$\psi = \sqrt{\frac{1}{5}}u_1 + \sqrt{\frac{1}{2}}u_2 + \sqrt{\frac{3}{10}}u_3$$

where u_n is the normalised state of the particle corresponding to an energy $E_n = n^2\epsilon$. What are the possible results of a measurement of the energy of the particle, and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the particle is 4.9ϵ.

[7 marks]

1.7) An electron is at rest in a magnetic field that has a field gradient $dB/dz = 15$ T m^{-1} along the direction of the field. What is the magnitude of the force on the electron?

[7 marks]

1.8) An electron is in the un-normalised spin state

$$\psi = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Normalise ψ. What is the probability that a measurement of the y component of the spin will correspond to the electron being in the spin down state?

[7 marks]

SECTION B – Answer TWO questions

2) A particle of mass m is completely confined in a one-dimensional 'box' of length a along the z axis. Write down the time-independent Schrödinger equation for the particle in the z dimension, and solve it to show that the allowed energy levels of the particle are

$$E = \frac{n^2 h^2}{8ma^2}, \quad n = 1, 2, 3, \ldots$$

[11 marks]

Derive an expression for the normalised wavefunction of the state of lowest energy, and show that the probability density $P(z)$ of the particle at the centre of the box is $2/a$.

[9 marks]

An electron is confined in a layer of a semiconductor that has a thickness $a = 2.0$ nm. Using the mass of the electron as given at the head of the paper, what is the energy of the lowest energy state?

[3 marks]

The atomic layers of the semiconductor crystal are separated by 0.2 nm. What is the fractional variation in that energy when the thickness decreases by one atomic layer from the mean?

[4 marks]

Describe briefly how variations in thickness on the atomic scale of a thin semiconductor layer that is buried inside another semiconductor could be measured.

[3 marks]

3) A particle of mass m that moves in one dimension (the x dimension) in a potential $V(x) = \frac{1}{2}m\omega^2x^2$ will perform harmonic oscillations about $x = 0$ with an angular fequency ω.

Write down the time-independent Schrödinger equation for the particle.

Given that the lowest energy state has the normalised eigenfunction

$$\psi_0(x) = (\alpha/\pi)^{1/4} \exp\left(-\alpha x^2/2\right),$$

where $\alpha = m\omega/\hbar$, show that the energy of this state is $E_0 = \hbar\omega/2$.

[12 marks]

Using the wavefunction $\psi_0(x)$, show that the mean square displacement of the particle is

$$\overline{x^2} = \frac{\hbar}{2m\omega}.$$

[10 marks]

An atom of hydrogen inside a crystal of silicon is observed to vibrate with an angular frequency of $\omega = 4.15 \times 10^{14}$ rad. s^{-1}.

What is the root mean square displacement of the hydrogen atom in its lowest energy state?

What is the fractional change in the angular frequency and in the root mean square displacement if the hydrogen atom is replaced by a deuterium atom (which has a nucleus of one proton and one neutron)?

[8 marks]

4) The operator for the square of the angular momentum operator, $\widehat{L^2}$, is given in spherical polar coordinates as

$$\widehat{L^2} = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right].$$

Calculate the eigenvalues of $\widehat{L^2}$ for the two spherical harmonics

$$Y_{00}(\theta,\phi) = \frac{1}{(4\pi)^{1/2}}, \quad Y_{10}(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta,$$

and verify that they are in agreement with the general expression $l(l+1)\hbar^2$.

[9 marks]

A molecule of HCl can be approximated as an atom of H orbiting around a much more massive atom of Cl. Write down an expression for the moment of inertia of a mass M rotating at a radius r around a fixed centre.

[2 marks]

According to theory, the energy of a rotating system is $E = L^2/2I$ where L^2 is the square of the angular momentum in the rotation and I is the moment of inertia, and radiation should be absorbed when the angular quantum number l increases by 1. Derive expressions, in terms of I, for the frequency of radiation absorbed by each of the transitions from $l = 0 \to l = 1$, $l = 1 \to l = 2$, $l = 2 \to l = 3$, and $l = 3 \to l = 4$.

In an experiment, radiation was observed to be absorbed by HCl at wavelengths of 238 and 161 μm. Calculate the frequencies of the radiation absorbed. Identify which two consecutive transitions are the most likely to fit the observations for HCl.

[9 marks]

Hence calculate the lengths of the H – Cl bond as derived from the two measured frequencies. Why is the length derived from the higher transition greater than that found from the lower transition?

[10 marks]

5) Classically, a hydrogen atom consists of one electron orbiting a proton in a circular orbit of radius r. Using classical mechanics, write down an equation that balances the Coulomb attraction between the electron and proton with the force required to maintain the circular motion.

Assuming that the radius is quantised as $r = n^2 a_0$, show that the frequency of the classical orbit is

$$f = \frac{e}{2\pi}\sqrt{\frac{1}{4\pi\epsilon_0 m}}\frac{1}{a_0^{3/2} n^3}.$$

[7 marks]

Derive a classical expression for the sum of the kinetic and potential energies of the electron–proton system. Hence show that in the limit of large n the frequency of the radiation emitted by a change $\Delta n = -1$ in the system is

$$\nu = \frac{e^2}{4\pi h\epsilon_0 n^3 a_0}.$$

[7 marks]

By equating the expressions for f and ν, show that the Bohr radius a_0 is

$$a_0 = \frac{h^2\epsilon_0}{m\pi e^2}.$$

[6 marks]

Given that the normalised wavefunction for the ground state of the H atom is

$$\psi(r) = \frac{2}{a_0^{3/2}}\exp\left(-\frac{r}{a_0}\right),$$

find the value of r at which the probability of finding the electron is maximum, and comment on the result.

[10 marks]