King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/2201 INTRODUCTORY QUANTUM MECHANICS

Summer 2002

Time allowed: THREE hours

Candidates must answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

Pauli matrices are given by

$$S_x = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

SECTION A — answer any SIX parts of this section

1.1) The normalized energy eigenfunction of the ground state of the hydrogen atom is

$$\psi(r) = Ce^{-r/a_0},$$

where a_0 is the Bohr radius and C is a constant. For this state, write down an expression for the probability of the electron lying within a spherical shell with radii r and $r + \mathrm{d}r$, and calculate the constant C.

Note:

$$\int_0^\infty e^{-\alpha r} r^n \, \mathrm{d}r = \frac{n!}{\alpha^{n+1}} ,$$

where the constant $\alpha > 0$ and the integer $n > -1$.

[7 marks]

1.2) Suppose that $\psi(x)$ and $\phi(x)$ are eigenstates of the observable A with real eigenvalues λ and μ, respectively. Show that if $\lambda \neq \mu$, then $\psi(x)$ and $\phi(x)$ are orthogonal.

[7 marks]

1.3) A quantum harmonic oscillator is in a state described by the normalized wave function

$$\psi(x) = \sqrt{\frac{1}{3}} u_0(x) + \sqrt{\frac{1}{6}} u_2(x) + \sqrt{\frac{1}{2}} u_4(x) ,$$

where $u_n(x)$ is the nth normalized energy eigenfunction of the oscillator corresponding to an eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$. What are the possible results of a measurement of the energy of this system and what are their relative probabilities? Using these probabilities, show that the expectation value of the energy of the oscillator is $\frac{17}{6}\hbar\omega$.

[7 marks]

1.4) Explain briefly what is meant by the *correspondence principle* and by the *complementarity principle.*

[7 marks]

1.5) The observables A and B are *compatible*. What does this imply about the eigenfunctions of the corresponding operators $\boldsymbol{A}$ and $\boldsymbol{B}$? Prove that $\boldsymbol{A}$ and $\boldsymbol{B}$ commute.

What are the physical implications of compatibility? Give one example of a compatible pair of observables.

[7 marks]

1.6) A quantum particle has mass m and moves freely in one dimension. For each real number k, the function $|k\rangle$ is defined to be

$$|k\rangle = \frac{1}{\sqrt{2\pi}} e^{ikx} \, .$$

Show that $|k\rangle$ is a (non-normalized) eigenfunction of the momentum operator with eigenvalue $\hbar k$. Prove that if ℓ and k are both real numbers

$$\langle \ell | k \rangle = \delta(k - \ell)$$

where $\delta(x)$ is the Dirac delta function.

Standard integral:

$$\int_{-\infty}^{\infty} e^{iax} \, \mathrm{d}x = 2\pi\delta(a) \, .$$

[7 marks]

1.7) An electron is in the unnormalized spin state

$$\psi = \begin{pmatrix} 3 \\ -4i \end{pmatrix} \, .$$

Normalize ψ and find the expectation value of the spin component $\boldsymbol{S}_y$ in this state. What is the probability that a measurement of $\boldsymbol{S}_y$ will correspond to the electron being in the spin-down state?

[7 marks]

1.8) Assume that the unit vector $\boldsymbol{n}$ lies in the $xz-$plane at an angle θ to the positive $z-$axis. Find the matrix representation of the component $\boldsymbol{S}_\theta$ of the spin operator $\boldsymbol{S}$ in the direction of $\boldsymbol{n}$. Calculate the eigenvalues of $\boldsymbol{S}_\theta$ and explain why the values obtained were only to be expected.

[7 marks]

SECTION B — answer TWO questions

2) Define an Hermitian operator.

[5 marks]

An Hermitian operator **A** corresponding to an observable A has normalized eigenfunctions u_1 and u_2 with corresponding eigenvalues a_1 and a_2. Similarly, the Hermitian operator **B** corresponding to observable B has normalized eigenfunctions v_1 and v_2 with corresponding eigenvalues b_1 and b_2. The eigenfunctions are related as follows

$$u_1 = (v_1 + 2v_2)/\sqrt{5}, \qquad u_2 = (2v_1 - v_2)/\sqrt{5}\,.$$

Suppose that when B is measured, the value b_1 is obtained. Following this measurement, A is measured followed by a second measurement of B. Show that the probability of obtaining b_2 is $\frac{8}{25}$.

[25 marks]

3) A beam of particles of mass m and energy E is incident from $x < 0$ upon a potential step at $x = 0$ of height $V_0\,(> E)$. Let

$$k^2 = \frac{2mE}{\hbar^2}\,, \qquad \kappa^2 = \frac{2m}{\hbar^2}(V_0 - E)\,.$$

The incident particles are represented by e^{ikx}. Calculate the reflection coefficient $\mathcal{R}$ and the transmission coefficient $\mathcal{T}$.

[15 marks]

Show that the amplitude of the reflected beam can be written as $e^{-2i\theta}$, where $\tan\theta = \kappa/k\,.$

[7 marks]

Hence show that the magnitude of the wave function in the region $x < 0$ is $2\cos(kx+\theta)$.

[8 marks]

4) In spherical polar coordinates (r, θ, ϕ), the components of the orbital angular momentum operator are given by

$$\boldsymbol{L}_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi})\,,$$
$$\boldsymbol{L}_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi})\,,$$
$$\boldsymbol{L}_z = -i\hbar\frac{\partial}{\partial\phi}\,.$$

State the commutation relations between pairs of angular momentum components and explain the physical implication of these relations.

[5 marks]

Show that the function

$$Y_{1,-1}(\theta,\phi) = \left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta\,\mathrm{e}^{-i\phi}$$

is an eigenfunction of $\boldsymbol{L}_z$ and determine the corresponding eigenvalue.

[5 marks]

Given that

$$\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = 4/3\,,$$

show that $Y_{1,-1}(\theta,\phi)$ is correctly normalized.

[5 marks]

The function $Y_{1,-1}(\theta,\phi)$ is one of the complete set of functions $Y_{\ell,m}(\theta,\phi)$.

(i) What is this set of functions called?

[2 marks]

(ii) The functions $Y_{\ell,m}$ are the simultaneous eigenfunctions of which two operators?

[2 marks]

(iii) Write down the eigenvalue equations for these two operators, expressing the eigenvalues in terms of ℓ, m and $\hbar$.

[4 marks]

(iv) What are the allowed values of ℓ and m?

[2 marks]

(v) Give a semi-classical argument which constrains the value of m for a given ℓ.

[5 marks]

5) Write down Schrödinger's equation of motion for a particle in a static potential and confined to the $x-$axis. Show how the $x-$ and $t-$variables can be separated and derive a particular solution for the time dependence.

[8 marks]

A quantum particle of mass m moves in one dimension subject to a potential that is zero in the region $-a \leq x \leq a$ and plus infinity elsewhere. The energy eigenvalues are $E_n = \hbar^2\pi^2 n^2/8ma^2$ for $n = 1, 2, 3, \ldots$ and the corresponding normalized eigenfunctions are

$$u_n(x) = \begin{cases} a^{-\frac{1}{2}} \cos(n\pi x/2a) \, , & n = 1, 3, 5, \ldots \\ a^{-\frac{1}{2}} \sin(n\pi x/2a) \, , & n = 2, 4, 6, \ldots \, . \end{cases}$$

Suppose that at $t = 0$, the particle is described by the normalized state function

$$\psi(x) = \frac{3}{5} u_3(x) + \frac{4}{5} u_4(x) \, .$$

(i) Write down the state function $\Psi(x, t)$ at time t;

[4 marks]

(ii) Calculate the probabilities of finding the particle at time t with the energies E_n $(n = 1, 2, 3, \ldots)$ and show they are the same as the corresponding probabilities at $t = 0$;

[8 marks]

(iii) Calculate the probability density $P(x, t)$ and hence determine how the probability density at the origin varies with t.

[10 marks]