King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/1400 Classical Mechanics and Special Relativity

Summer 2003

Time allowed: THREE Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

In all this exam paper, t denotes the time and a dot over a letter denotes a derivative with respect to time

SECTION A – Answer SIX parts of this section

1.1) $\vec{e}_r$ and $\vec{e}_\theta$ are the unit vectors in a plane with polar coordinates (r, θ). Given that $\dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta$ and $\dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r$, show that the acceleration due to motion in a circle with radius R and constant angular velocity ω is $\vec{a} = -\omega^2 R\ \vec{e}_r$.

[7 marks]

1.2) Define and describe the resonance that occurs in certain conditions when an oscillator is forced to oscillate by an external operator.

[7 marks]

1.3) A point particle is subject to a central force $\vec{f}$. Define its angular momentum $\vec{\mathcal{L}}$ and show that it is a constant of the motion. Hence deduce that the trajectory is planar.

[7 marks]

1.4) Describe the different trajectories of Kepler's mechanics and give the energies associated with each case.

[7 marks]

1.5) The differential equation which describes the electric oscillations in a circuit having capacitance C and inductance L is $L\ddot{q} + q/C = 0$, where q is the electric charge. State the analogous mechanical equation and deduce the frequency of the electric oscillations.

[7 marks]

1.6) The moment of inertia of a solid homogeneous ball of radius R and mass M with respect to a diameter is $2MR^2/5$. Use an appropriate theorem to derive an expression for the moment of inertia with respect to an axis tangent to the ball.

[7 marks]

1.7) Define and inertial frame and explain the origin of the inertial forces that are observed in a non-inertial frame.

[7 marks]

1.8) Sketch a Minkowski diagram for a 1+1 dimensional space-time (t, x) centered on an event O and indicate the region which contains events that can be influenced by O.

[7 marks]

SECTION B – Answer TWO questions

2) A solid homogeneous cylinder Σ of mass M and radius r rolls inside another fixed cylinder of radius $R > r$, their axes of symmetry being parallel. The angular velocity of Σ in its centre of mass frame is denoted by $\dot{\phi}$ and its centre of mass has angular velocity $\dot{\theta}$ in the frame of the fixed cylinder ($\dot{\phi}$ and $\dot{\theta}$ are defined with the same sign).

a) Show that the moment of inertia of Σ relative to its axis of symmetry is $Mr^2/2$.

[5 marks]

b) Show that, if Σ does not slip, the angular velocities are such that $R\dot{\theta} = r\dot{\phi}$.

[5 marks]

c) Show that the kinetic energy of Σ is $E_k = M(\dot{\theta})^2 \left(3R^2/4 + r^2/2 - rR\right)$.

[7 marks]

d) ¿From the total energy of Σ, show that the differential equation satisfied by θ is $\ddot{\theta}[(R-r)^2 + R^2/2] + g(R-r)\sin\theta = 0$.

[8 marks]

e) Consider the case where θ is small and thus $\sin\theta \simeq \theta$. Derive an expression for the angular frequency of small oscillations about the lowest position.

[5 marks]

3) A seismograph consists of a mass m suspended from a vertical spring of force constant k and damping coefficient λ. The seismograph rest frame is not inertial but oscillates with amplitude $A\cos(\Omega t)$ in the inertial frame where the experiment takes place. The coordinate in the seismograph rest frame is z, with origin $z = 0$ at the equilibrium position of the mass m. The oscillations are observed in the non-inertial rest frame of the seismograph.

a) Explain the origin of the different forces in the equation of motion

$$-kz - \lambda\dot{z} + mA\Omega^2\cos(\Omega t) = m\ddot{z}.$$

[7 marks]

b) The general solution for weak damping is

$$z(t) = z_0 \exp\left(-\frac{t}{\tau}\right)\cos(\omega t + \phi_0) + Z\cos(\Omega t + \phi),$$

where z_0 and ϕ_0 are constants of integration. Use the solution in the absence of excitation, i.e. $A = 0$, $Z = 0$, to obtain expressions for the proper angular frequency ω and the characteristic time τ, as functions of k, λ and m.

[7 marks]

c) For $t >> \tau$, the transient motion may be neglected and $z(t) \simeq Z\cos(\Omega t + \phi)$. Show that

$$Z = \frac{A\ m\ \Omega^2}{\sqrt{\lambda^2\Omega^2 + (k - m\Omega^2)^2}},$$

and

$$\tan\phi = \frac{\lambda\Omega}{m\Omega^2 - k}.$$

[10 marks]

d) Sketch the curves $Z(\Omega)$ and $\phi(\Omega)$ and discuss their shapes in the limit when $\lambda \to 0$.

[6 marks]

4) A projectile of mass m is launched from the Earth (radius R, mass M) which rotates with angular velocity ω in an inertial frame. The projectile is launched from the equator, vertically in the rotating frame of the Earth, with velocity v_0. It reaches an altitude h (measured from the surface of the Earth) with zero radial velocity and angular velocity Ω.

a) Derive an expression for the angular momentum of the projectile with respect to the centre of the Earth, in polar coordinates (r, θ).

[5 marks]

b) Use the conservation of angular momentum to deduce that $(R+h)^2\Omega = R^2\omega$.

[5 marks]

c) Form an expression for the total energy of the projectile in polar coordinates.

[5 marks]

d) Use the conservation of energy to infer that h satisfies

$$v_0^2 = \frac{2GMh}{R(R+h)} - \omega^2 R^2 \frac{h^2 + 2Rh}{(R+h)^2},$$

where G is the gravitational constant.

[5 marks]

e) Show that the velocity of a satellite on a circular orbit at the altitude h is

$$v = \sqrt{\frac{GM}{(R+h)}}.$$

[5 marks]

f) Hence derive an expression for the additional velocity that must be given to the projectile when it reaches the altitude h in order for it to attain a circular orbit as a satellite.

[5 marks]

5) A pulse of light with frequency ν_0 is emitted in an inertial frame S' moving at velocity v with respect to another inertial frame S. An observer O at rest in S sees the light generator approaching on the axis (Ox) and measures the frequency ν. The Lorentz transformations are

$$ct' = \gamma\left(ct - \frac{v}{c}x\right) \qquad x' = \gamma(x - vt),$$

where c is the speed of light, $\gamma = (1 - v^2/c^2)^{-1/2}$ and the coordinates (t, x), (t', x') refer to S, S' respectively. The light is characterized by its phase

$\phi = 2\pi\nu_0(t - x/c)$ in S and $\phi' = 2\pi\nu(t' - x'/c)$ in S'.

a) Give reasons to explain why $\phi' = \phi$, i.e. the phase is independent of the inertial frame.

[5 marks]

b) Show that

$$\left(t' - \frac{x'}{c}\right) = \gamma\left(1 + \frac{v}{c}\right)\left(t - \frac{x}{c}\right).$$

[5 marks]

c) Hence use phase invariance to deduce that

$$\nu = \nu_0\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

[5 marks]

d) What would be the relation between ν and ν_0 if the light generator were receding from, rather than approaching the observer?

[5 marks]

e) Determine the numerical value of ν/ν_0 (in the case of question c) when

$v = 0.8c$.

[5 marks]

f) Sketch the Minkowski diagram corresponding to a ray of light emitted in S' and observed in S.

[5 marks]