King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/1210 Mathematical Methods in Physics I

Summer 1999

Time allowed: 3 Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) Show by substitution that $x = A\cos(\omega t + \gamma)$ is a solution to the harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

and find A and γ if at $t = 0$ the oscillator is stationary at $x = A_0$.

[7 marks]

1.2) The temperature T of a body increases at a rate $dT/dt = k(T_o - T)$ where T_o is the constant temperature of the surroundings. Show that $T = T_o - C\exp(-kt)$ where C is a constant.

[7 marks]

1.3) Given that

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

is an eigenvector of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

what is the corresponding eigenvalue?

[7 marks]

1.4) Show that the simultaneous equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

are satisfied by $x = \Delta_1/\Delta$ where Δ_1 and Δ are the determinants

$$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

[7 marks]

1.5) Is the vector field $\mathbf{E} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ irrotational or solenoidal or neither?

[7 marks]

1.6) Given the scalar field $\phi = -1/r^3$ where $r = (x^2 + y^2 + z^2)^{1/2}$, calculate grad ϕ.

[7 marks]

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1.7) Given the vector field $\mathbf{E} = z\mathbf{i} + 2y\mathbf{j} + x\mathbf{k}$ calculate the line integral $\int_C \mathbf{E}.d\mathbf{r}$ where C is the arc of the circle $x^2 + y^2 = 1$ in the plane $z = 2$, from the point (1,0,2) to (0,1,2).

[7 marks]

1.8) The Fourier series representation of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1\,, \\ 0 & \text{if } 1 < x < 2\,, \end{cases}$$

is

$$F(f(x)) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sin n\pi x}{n}\,.$$

What is the expected value of F at $x = 1$, and does this agree with the value given by the series? Find the sum of the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\,.$$

[7 marks]

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SECTION B – Answer TWO questions

2) An isotope of thorium decays to radium which in turn decays to radon. If at time $t = 0$ a sample contains N_0 of the unstable thorium nuclei, show that the number n of radium nuclei obeys

$$\frac{dn}{dt} + \lambda_2 n = \lambda_1 N_0 \exp(-\lambda_1 t)$$

where $\lambda_1,\ \lambda_2$ are respectively the decay rates of thorium and radium.

[7 marks]

Hence show that if $\lambda_1 \neq \lambda_2$ and $n = 0$ at $t = 0$

$$n = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right),$$

[13 marks]

and show that the maximum number of radium nuclei occurs at

$$t_m = \frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{\lambda_2}{\lambda_1}\right).$$

[10 marks]

[*You may assume that the solution to* $dy/dx + Py = Q$ *where* P *and* Q *are functions of* x *is*

$$y = e^{-I}\int Qe^{I}dx + ce^{-I}$$

where c *is a constant and* $I = \int Pdx$.]

3) The position x of a particle varies with time t according to

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$$

where ω and k are constants.

Show that a solution for x is

$$x = e^{-kt}\left(Ae^{i\beta t} + Be^{-i\beta t}\right)$$

where A and B are constants and $\beta^2 = \omega^2 - k^2$.

[10 marks]

Show that if $x = 0$ at $t = 0$ then

$$x = 2iAe^{-kt}\sin(\beta t).$$

[10 marks]

Assuming $k^2 << \omega^2$, what is the ratio of the amplitudes of successive oscillations with $x > 0$?

[10 marks]

4) Calculate div$\mathbf{A}$ and curl $\mathbf{A}$ when $\mathbf{A} = 2x\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

[5 marks]

The transformation from Cartesian coordinates (x, y, z) to spherical polar coordinates (r, θ, ϕ) is given by

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta\,.$$

Show that the Jacobian of the transformation is $r^2\sin\theta$.

[6 marks]

Stokes' theorem states that

$$\int_S \mathrm{curl}\,\mathbf{A}.d\mathbf{S} = \int_C \mathbf{A}.d\mathbf{r}\ ,$$

where $\mathbf{A}$ is a vector field and C is the boundary of a regular open surface S. Verify Stokes' theorem directly for the given vector field $\mathbf{A}$ when S is the surface of the upper half of the sphere of radius $r = R$ and C is the circle in the (x, y)-plane of radius R.

[13 marks]

Use Gauss' theorem to show that

$$\int_{S'} \mathbf{A}.d\mathbf{S} = 2\pi R^3\,,$$

where S' is the closed surface given by S (above) and the (x, y)-plane.

[6 marks]

5) The Fourier cosine series of an even function $f(x)$ in the range $-T/2 \leq x \leq T/2$ has the form

$$F(f(x)) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n \cos\left(\frac{2n\pi x}{T}\right)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2n\pi x/T)dx\,,$$

for $n = 0, 1, 2 \ldots$.
Show that the Fourier series for the function

$$f(x) = |x|, \ -T/2 < x < T/2$$

is

$$F(f(x)) = \frac{T}{4} + \frac{T}{\pi^2}\sum_{n\geq 1}\frac{((-1)^n - 1)}{n^2}\cos(2n\pi x/T)\,.$$

[16 marks]

Sketch the Fourier series representation of $f(x)$ in the interval $-\frac{3}{2}T \leq x \leq \frac{3}{2}T$. Add to your sketch the function obtained by including only the first two terms of the Fourier series.

[7 marks]

By considering the value of the Fourier series at $x = T/2$, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}\,.$$

[7 marks]