King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/1210 Mathematical Methods in Physics I

Summer 1998

Time allowed: 3 Hours

Candidates should answer SIX parts of SECTION A,
and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper.
Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) A ball falling through a vacuum in a uniform gravitational field has a height z which obeys the equation

$$\frac{d^2z}{dt^2} = -g$$

where g is the acceleration caused by gravity. At time $t = 0$ a ball is thrown vertically upwards from $z = 0$ with an initial speed $\sqrt{2g}$. Show that the height reached at the time $t = \sqrt{2/g}$ is $z = 1$.

[7 marks]

1.2) When a gas expands adiabatically the volume V and pressure P obey

$$\frac{C_p}{V} + \frac{C_v}{P}\frac{dP}{dV} = 0$$

where C_p and C_v are constants. Show that in the adiabatic expansion PV^n is a constant with $n = C_p/C_v$.

[7 marks]

1.3) The reflection of a point (x, y, z) in the $x = 0$ plane is described by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the rotation of a point about the z axis through an angle α is given by the matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show by matrix multiplication that a reflection followed by a rotation does not produce the same result as rotation followed by reflection.

[7 marks]

1.4) Find the two eigenvalues k of the eigenvalue equation

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} x \\ y \end{pmatrix}.$$

[7 marks]

1.5) Calculate div**A** and curl**A** when **A** is the vector $y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

[7 marks]

1.6) The height of a hill above the (x, y)-plane is given by the function

$$\phi = \frac{1}{(1 + x^2 + y^2)}.$$

What is the slope of the hill at the point $(1, 0)$ in (a) the direction of the x-axis, and (b) the direction of the y-axis?

[7 marks]

1.7) Calculate the value of the line integral

$$\int_C \mathbf{r} d\mathbf{r}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and C is (i) the line $y = x$ from the point (0,0) to (1,1) and, (ii) the two straight lines from (0,0) to (1,0) and from (1,0) to (1,1).

[7 marks]

1.8) The Fourier series representation of the function $f(x) = x$, when one period of the Fourier series lies in the interval $[-a/2, a/2]$, is

$$F(f(x)) = \frac{a}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2\pi n x}{a} \quad .$$

What is the value of the Fourier series representation at $x = a/2$, and is this the value you would expect?

[7 marks]

SECTION B – Answer TWO questions

2) A mass m hangs stationary on the end of a weightless spring. When the mass is pulled down through a small distance and then released, its position varies with time t according to

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + cx = 0$$

where x is the extension of the spring measured from the equilibrium position, and r and c are constants. The mass is adjusted so that

$$r^2 = 4mc.$$

Show that the auxiliary equation method gives one solution for x of

$$x = A_1 e^{-rt/2m}$$

where A_1 is an arbitrary constant.

[5 marks]

Verify that a second solution is $x = A_2 t e^{-rt/2m}$, where A_2 is another arbitrary constant.

[10 marks]

The mass is initially at rest in the equilibrium position $x = 0$, but at $t = 0$ it is suddenly pulled down at a speed $dx/dt = u$. Show that at later times the position of the mass is given by

$$x = ute^{-rt/2m}.$$

[7 marks]

Show that the maximum value of x occurs at a time

$$t = 2m/r$$

and derive the value of that maximum value of x.

[8 marks]

Note: You may assume that the solution to

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c = 0$$

is

$$x = A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

where m_1 and m_2 are the roots of the auxiliary equation $am^2 + bm + c = 0$.

3) A radioactive species A decays into a second species B, which in turn is unstable. At time $t = 0$ there are N_0 atoms of A, and none of B. The rate at which the nuclei of A decay is $k_A N_A$ where k_A is a constant. The rate of decay of species B is $k_B N_B$ where k_B is a constant and N_B is the number of atoms of B.

Show that N_B obeys

$$\frac{dN_B}{dt} + k_B N_B = k_A N_0 \exp(-k_A t).$$

[7 marks]

Hence show that if $k_A \neq k_B$ and $N_B = 0$ at $t = 0$

$$N_B = \frac{k_A N_0}{k_B - k_A} \left(e^{-k_A t} - e^{-k_B t}\right),$$

[8 marks]

and that the maximum number of B atoms occurs at the time

$$t_m = \frac{1}{k_B - k_A} \ln\left(\frac{k_B}{k_A}\right).$$

[15 marks]

Note: You may assume that the solution to $dy/dx + Py = Q$ where P and Q are functions of x is

$$y = e^{-I} \int Q e^I dx + c e^{-I}$$

where c is a constant and $I = \int P dx$.

4) Calculate div **A** when $\mathbf{A} = \sqrt{x^2+y^2}(x\mathbf{i}+y\mathbf{j})$.

[7 marks]

The transformation from Cartesian coordinates (x, y, z) to cylindrical coordinates (r, θ, z') is given by

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = z'\ .$$

Show that the Jacobian of the transformation is r.

[7 marks]

The divergence theorem states that

$$\int_V \text{div}\,\mathbf{A}\,dv = \int_S \mathbf{A}.d\mathbf{S}\ ,$$

where $\mathbf{A}$ is a vector field and V is the volume bounded by a simple closed surface S. Verify the divergence theorem directly for the given vector field $\mathbf{A}$, when the volume V is the cylinder $x^2+y^2 \leq 4,\ 0 \leq z \leq 5$, by using cylindrical coordinates to perform the volume and surface integrals. You are given that the value of both integrals is 80π.

[16 marks]

5) Expand $f(x) = |x|$ as a cosine Fourier series in the range $-T/2 \leq x \leq T/2$. The general form for such a series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right)$$

where $a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2n\pi x/T)dx$, for $n = 0, 1, 2\ldots$, and T is the period.

[16 marks]By considering the value of the Fourier series at $x = 0$, show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}\ .$$

What is the value of the Fourier series at $x = T/2$?

[7 marks]

Sketch the Fourier series representation of $f(x)$ in the interval $-\frac{3}{2}T \leq x \leq \frac{3}{2}T$. Add to your sketch the function obtained by including only the first two terms of the Fourier series.

[7 marks]