King's College London

UNIVERSITY OF LONDON

B.Sc. EXAMINATION

CP/1210 Mathematical Methods in Physics I

Summer 1997

Time allowed: 3 Hours

Candidates should answer SIX parts of SECTION A, and TWO questions from SECTION B.

Separate answer books must be used for each Section of the paper.

The approximate mark for each part of a question is indicated in square brackets.

You must not use your own calculator for this paper. Where necessary, a College calculator will have been supplied.

TURN OVER WHEN INSTRUCTED

SECTION A – Answer SIX parts of this section

1.1) A body falling through a vacuum in a uniform gravitational field has a height z which obeys the equation

$$\frac{d^2z}{dt^2} = -g$$

where g is the acceleration caused by gravity. At time $t = 0$ both the height and the speed are zero. Prove that at $t > 0$, $z = -gt^2/2$.

[7 marks]

1.2) A string is stretched along the x axis with its ends fixed at $x = 0$ and $x = l$. When it is plucked, the displacement y of the string obeys

$$\frac{d^2y}{dx^2} = -k^2 y$$

where k is a constant. Show by substitution that $y = a \sin kx + b \cos kx$ where a and b are constants.

Why is $b = 0$, and what are the possible values of k.

[7 marks]

1.3) The number N of radioactive nuclei changes, from an original value N_0, at the rate $dN/dt = -kN$ where k is a constant. Solve for N and show that the radioactive half-life is $\tau = (\ln 2)/k$.

[7 marks]

1.4) Show that the eigenvalue equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = k \begin{pmatrix} x \\ y \end{pmatrix},$$

where k is the eigenvalue associated with the vector $\begin{pmatrix} x \\ y \end{pmatrix}$, has eigenvectors with x and y components in the ratios

$$\frac{x}{y} = \frac{b}{k-a}, \quad \text{and} \quad \frac{k-d}{c}.$$

[7 marks]

1.5) A vector field $\mathbf{F}$ has the value

$$\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/r^3$$

at the point $(x,\ y,\ z)$, a distance r from the origin. Show that $\nabla.\mathbf{F} = 0$ at all points $r \neq 0$.

[7 marks]

1.6) Show that the Jacobean for the transformation $u = e^x \cos y$, $v = e^{-x} \sin y$ defined by

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is $\cos^2 y - \sin^2 y$.

[7 marks]

1.7) The Fourier series for a smooth even function $f(x)$ of period T is given by

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{2\pi n x}{T}\right)$$

where the coefficients b_n are independent of x for all n. Show that

$$\int_{-T/2}^{T/2} f(x)dx = Tb_0.$$

[7 marks]

1.8) Show that the value of the double integral

$$\int_1^2 dx \int_0^x \frac{1}{x^2} dy$$

is $\ln 2$.

[7 marks]

SECTION B – Answer TWO questions

2) A weight hanging on a spring is driven by an applied sinusoidal force so that its displacement y from its equilibrium position obeys

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + \omega^2 y = F \sin \omega_1 t$$

where b and ω are constants. By considering y as the imaginary part of $\exp(i\omega_1 t)$, show that the steady state solution is

$$y = F \sin(\omega_1 t - \phi) \Big/ \sqrt{(\omega^2 - \omega_1^2)^2 + b^2\omega_1^2},$$

and show that the phase angle $\phi = \tan^{-1}\left(b\omega_1/(\omega^2 - \omega_1^2)\right)$.

[10 marks]

Hence show that when the driving frequency ω_1 is varied

a) the maximum amplitude of y occurs at

$$\omega_1^2 = \omega^2 - \tfrac{1}{2}b^2,$$

[10 marks]

b) the maximum amplitude of the speed occurs at

$$\omega_1 = \omega.$$

[10 marks]

3) An inductor of inductance L, a resistor of resistance R and a capacitor of capacitance C are connected in series. The charge q on the capacitor varies with time t according to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0.$$

Prove that a solution for q is

$$q = e^{-\alpha t}\left(A_1 e^{i\beta t} + A_2 e^{-i\beta t}\right)$$

where $\alpha = R/2L$, $\beta^2 = (4L/C) - R^2$, and A_1 and A_2 are constants.

[8 marks]

When the circuit is connected, at $t = 0$, the current flowing in it is zero and $q = q_0$. Show that

$$q = q_0 e^{-\alpha t}\left[\cos(\beta t) + \frac{\alpha}{\beta}\sin(\beta t)\right].$$

[12 marks]

Assuming that $R^2 < 4L/C$ and $R << 2L$, what is the time period of the oscillations of q?

Show that the ratio of the amplitudes of successive oscillations separated by one time period is $\exp(-\pi R/L\beta)$.

[10 marks]

You may assume that the solution to

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c = 0$$

is

$$x = A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

where m_1 and m_2 are the roots of the auxiliary equation $am^2 + bm + c = 0$.

4) The function $f(x)$ is defined as

$$f(x) = 1, \quad 0 < x < \pi$$

$$f(x) = 0, \quad \pi < x < 2\pi.$$

Sketch $f(x)$ in the range $0 < x < 2\pi$.

[3 marks]

$f(x)$ is expanded in a Fourier sine series,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx.$$

Show that $a_0 = 1$, that $b_n = 0$ when n is even, and $b_n = 2/(n\pi)$ when n is odd.

[10 marks]

Sketch, in the range $0 < x < 2\pi$, the first three non-zero terms of the expansion.

[7 marks]

By considering the value of $f(x)$ at $x = \pi/2$, use the Fourier series to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots = \frac{\pi}{4}.$$

[10 marks]

5) A point P has Cartesian coordinates x, y, z. Show that a rotation through an angle of $+\alpha$ about the z-axis changes the coordinates of P to new values X, Y, Z, where:

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[10 marks]

Show that the transpose R^T of R is the inverse of R.

[3 marks]

Write down the matrix which describes the effect on x, y, z of a reflection in the $z = 0$ plane.

[3 marks]

P is first rotated through an angle of $-\alpha$ about the z-axis and is then reflected in the $z = 0$ plane. Show, by calculating its final position, that it is still at the same distance, $\sqrt{x^2 + y^2 + z^2}$, from the origin.

[8 marks]

Show that if the reflection took place before the rotation, the final position would be the same (the operations commute).

[6 marks]