Appendix C

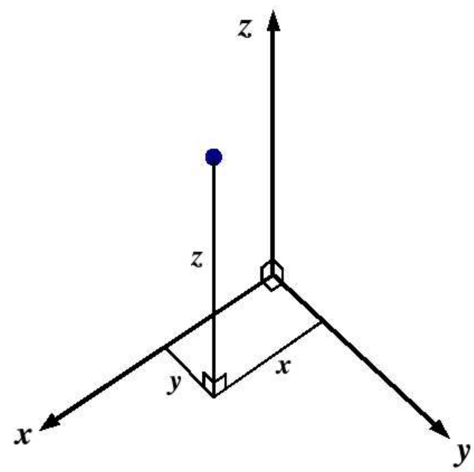
Standard Results for Vectors

C1. Summary

This appendix summarises some basic results relating to vectors, in particular for different coordinate systems. Many of these will be familiar, but it is worth stating them in detail.

C2. Vector Identities for Cartesian Coordinate Systems

Consider a Cartesian coordinate system (x, y, z) as shown in the figure.



In the following, $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$ are unit vectors in the x, y and z directions. Vectors $\mathbf{A}$ and $\mathbf{B}$ are resolved into their components as

$$\mathbf{A} \;=\; A_x\,\hat{\mathbf{\imath}} + A_y\,\hat{\mathbf{\jmath}} + A_z\,\hat{\mathbf{k}} \qquad \text{and} \qquad \mathbf{B} \;=\; B_x\,\hat{\mathbf{\imath}} + B_y\,\hat{\mathbf{\jmath}} + B_z\,\hat{\mathbf{k}} \;.$$

The following results apply to *any* vectors $\mathbf{A}$ and $\mathbf{B}$.
The dot product (scalar product) is

$$\mathbf{A\,.\,B} \;=\; A_xB_x + A_yB_y + A_zB_z$$

The cross product (vector product) is

$$\mathbf{A}\times\mathbf{B} \;=\; (A_yB_z - A_zB_y)\,\hat{\mathbf{\imath}} + (A_zB_x - A_xB_z)\,\hat{\mathbf{\jmath}} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}$$

The gradient of a scalar field f is

$$\boldsymbol{\nabla} f \;=\; \hat{\mathbf{\imath}}\,\frac{\partial f}{\partial x} + \hat{\mathbf{\jmath}}\,\frac{\partial f}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial f}{\partial z} \;.$$

The divergence of a vector is

$$\boldsymbol{\nabla}\,.\,\mathbf{A} \;=\; \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \;.$$

The curl of a vector is

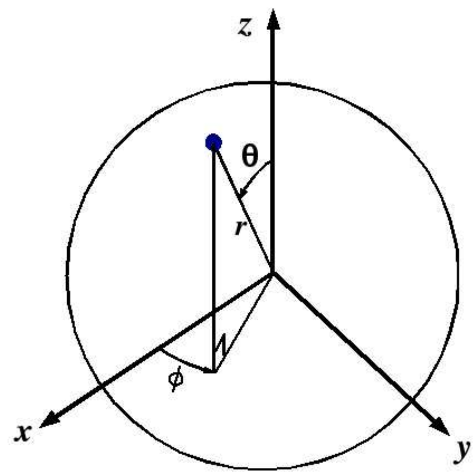
$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}} \ .$$

The Laplacian of a scalar function f is

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \ .$$

C3. Vector Identities for Spherical Polar Coordinate Systems

Consider a spherical polar coordinate system (r, θ, ϕ) as shown in the figure.
Note the definition of the angles θ and ϕ here: these definitions are used in the results below. Some authors choose to switch the definitions of θ and ϕ. We define θ and ϕ in this way here because the angle ϕ can be compared directly with the angle ϕ in the cylindrical coordinate system. The Cartesian (x, y, z) axes are also shown for comparison.



In the following, $\hat{\mathbf{e}}_{\boldsymbol{r}}$, $\hat{\mathbf{e}}_{\boldsymbol{\theta}}$ and $\hat{\mathbf{e}}_{\boldsymbol{\phi}}$ are unit vectors in the r, θ and ϕ directions. Vectors $\mathbf{A}$ and $\mathbf{B}$ are resolved into their components as

$$\begin{aligned} \mathbf{A} &= A_r\,\hat{\mathbf{e}}_{\boldsymbol{r}} + A_\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}} + A_\phi\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} \\ \text{and} \quad \mathbf{B} &= B_r\,\hat{\mathbf{e}}_{\boldsymbol{r}} + B_\theta\,\hat{\mathbf{e}}_{\boldsymbol{\theta}} + B_\phi\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} \end{aligned}$$

The following results apply to any vectors $\mathbf{A}$ and $\mathbf{B}$.
The dot product (scalar product) is

$$\mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

The cross product (vector product) is

$$\mathbf{A} \times \mathbf{B} = (A_\theta B_\phi - A_\phi B_\theta)\,\hat{\mathbf{e}}_{\boldsymbol{r}} + (A_\phi B_r - A_r B_\phi)\,\hat{\mathbf{e}}_{\boldsymbol{\theta}} + (A_r B_\theta - A_\theta B_r)\,\hat{\mathbf{e}}_{\boldsymbol{\phi}}$$

The gradient of a scalar field f is

$$\nabla f = \hat{\mathbf{e}}_{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\mathbf{e}}_{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\mathbf{e}}_{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi} \ .$$

The divergence of a vector is

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} \ .$$

The curl of a vector is

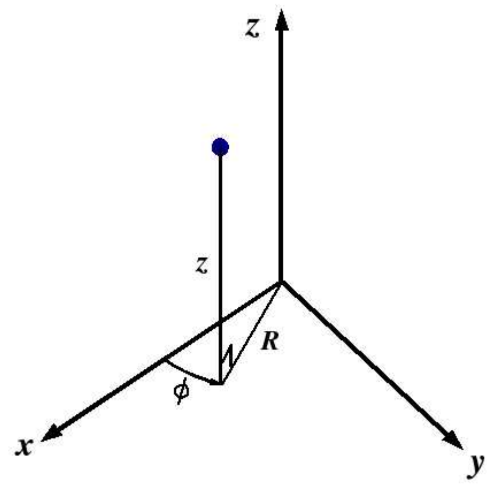
$$\nabla \times \mathbf{A} = \hat{\mathbf{e}}_{\boldsymbol{r}} \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right) + \\ + \hat{\mathbf{e}}_{\boldsymbol{\theta}} \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\mathbf{e}}_{\boldsymbol{\phi}} \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)$$

The Laplacian of a scalar function f is

$$\nabla^2 f \equiv \nabla\mathbf{.}(\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

C4. Vector Identities for Cylindrical Polar Coordinate Systems

Consider a cylindrical polar coordinate system (R, ϕ, z) as shown in the figure.
The Cartesian (x, y, z) axes are also shown for comparison.
The coordinate angle is called ϕ here, although some authors prefer to call it θ.



In the following, $\hat{\mathbf{e}}_{\mathbf{R}}$, $\hat{\mathbf{e}}_{\boldsymbol{\phi}}$ and $\hat{\mathbf{e}}_{\mathbf{z}}$ are unit vectors in the R, ϕ and z directions. Vectors $\mathbf{A}$ and $\mathbf{B}$ are resolved into their components as

$$\begin{aligned} \mathbf{A} &= A_R\,\hat{\mathbf{e}}_{\mathbf{R}} + A_\phi\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} + A_z\,\hat{\mathbf{e}}_{\mathbf{z}} \\ \text{and} \quad \mathbf{B} &= B_R\,\hat{\mathbf{e}}_{\mathbf{R}} + B_\phi\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} + B_z\,\hat{\mathbf{e}}_{\mathbf{z}} \end{aligned}$$

The following results apply to any vectors $\mathbf{A}$ and $\mathbf{B}$.
The dot product (scalar product) is

$$\mathbf{A}\mathbf{.}\mathbf{B} = A_R B_R + A_\phi B_\phi + A_z B_z \ .$$

The cross product (vector product) is

$$\mathbf{A} \times \mathbf{B} = (A_\phi B_z - A_z B_\phi)\,\hat{\mathbf{e}}_{\mathbf{R}} + (A_z B_R - A_R B_z)\,\hat{\mathbf{e}}_{\boldsymbol{\phi}} + (A_R B_\phi - A_\phi B_R)\,\hat{\mathbf{e}}_{\mathbf{z}} \ .$$

The gradient of a scalar field f is

$$\nabla f = \hat{\mathbf{e}}_{\mathbf{R}}\frac{\partial f}{\partial R} + \hat{\mathbf{e}}_{\boldsymbol{\phi}}\frac{1}{R}\frac{\partial f}{\partial\phi} + \hat{\mathbf{e}}_{\mathbf{z}}\frac{\partial f}{\partial z} \ .$$

The divergence of a vector is

$$\boldsymbol{\nabla}\boldsymbol{.}\,\mathbf{A} \;=\; \frac{1}{R}\frac{\partial}{\partial R}(RA_R) \;+\; \frac{1}{R}\frac{\partial A_\phi}{\partial \phi} \;+\; \frac{\partial A_z}{\partial z} \;\;.$$

The curl of a vector is

$$\boldsymbol{\nabla}\times\mathbf{A} \;=\; \hat{\mathbf{e}}_{\mathbf{R}}\left(\frac{1}{R}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) \;+\; \hat{\mathbf{e}}_{\boldsymbol{\phi}}\left(\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R}\right) \;+\; \hat{\mathbf{e}}_{\mathbf{z}}\,\frac{1}{R}\left(\frac{\partial}{\partial R}(RA_\phi) - \frac{\partial A_R}{\partial \phi}\right) \;\;.$$

The Laplacian of a scalar field f is

$$\nabla^2 f \;\equiv\; \boldsymbol{\nabla}\boldsymbol{.}\,(\boldsymbol{\nabla} f) \;=\; \nabla^2 f \;\equiv\; \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) \;+\; \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} \;+\; \frac{\partial^2 f}{\partial z^2} \;\;.$$

C5. Position Vectors, Velocity Vectors and Acceleration Vectors

In a Cartesian coordinate system (x, y, z) with unit vectors $\hat{\mathbf{\imath}}$, $\hat{\mathbf{\jmath}}$ and $\hat{\mathbf{k}}$, the position vector $\mathbf{r}$, the velocity vector $\mathbf{v}$ and the acceleration vector $\mathbf{a}$ are

$$\begin{aligned}
\mathbf{r} \;&=\; x\,\hat{\mathbf{\imath}} \;+\; y\,\hat{\mathbf{\jmath}} \;+\; z\,\hat{\mathbf{k}} \\
\mathbf{v} \;&=\; \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{\mathbf{\imath}} \;+\; \frac{\mathrm{d}y}{\mathrm{d}t}\,\hat{\mathbf{\jmath}} \;+\; \frac{\mathrm{d}z}{\mathrm{d}t}\,\hat{\mathbf{k}} \\
\mathbf{a} \;&=\; \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\hat{\mathbf{\imath}} \;+\; \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\,\hat{\mathbf{\jmath}} \;+\; \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\,\hat{\mathbf{k}}
\end{aligned}$$

for any position, velocity and acceleration.
(Note that these expressions apply whatever the velocity and acceleration are, and whatever forces drive the acceleration.)

In a spherical polar coordinate system (r, θ, ϕ) with unit vectors $\hat{\mathbf{e}}_{\boldsymbol{r}}$, $\hat{\mathbf{e}}_{\boldsymbol{\theta}}$ and $\hat{\mathbf{e}}_{\boldsymbol{\phi}}$, we have

$$\begin{aligned}
\mathbf{r} \;&=\; r\,\hat{\mathbf{e}}_{\boldsymbol{r}} \\
\mathbf{v} \;&=\; \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \;=\; \hat{\mathbf{e}}_{\boldsymbol{r}}\,\frac{\mathrm{d}r}{\mathrm{d}t} \;+\; \hat{\mathbf{e}}_{\boldsymbol{\theta}}\,r\,\frac{\mathrm{d}\theta}{\mathrm{d}t} \;+\; \hat{\mathbf{e}}_{\boldsymbol{\phi}}\,r\sin\theta\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \\
\mathbf{a} \;&=\; \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \;=\; \hat{\mathbf{e}}_{\boldsymbol{r}}\left(\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 - r\sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right) \\
&\qquad +\; \hat{\mathbf{e}}_{\boldsymbol{\theta}}\left(2\,\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\theta}{\mathrm{d}t} \;+\; r\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \;-\; r\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right) \\
&\qquad +\; \hat{\mathbf{e}}_{\boldsymbol{\phi}}\left(r\sin\theta\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \;+\; 2\sin\theta\,\frac{\mathrm{d}r}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} \;+\; 2r\cos\theta\,\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)
\end{aligned}$$

for any position, velocity and acceleration.
(Note that $\hat{\mathbf{e}}_{\boldsymbol{r}}, \hat{\mathbf{e}}_{\boldsymbol{\theta}}$ and $\hat{\mathbf{e}}_{\boldsymbol{\phi}}$ are unit vectors in the r, θ and ϕ directions at any time and that they change direction as the particle moves. These expressions for $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ apply whatever the velocity and acceleration are, and whatever forces drive the acceleration.)

In a cylindrical coordinate system (R, ϕ, z) with unit vectors $\hat{\mathbf{e}}_{\mathbf{R}}$, $\hat{\mathbf{e}}_{\boldsymbol{\phi}}$ and $\hat{\mathbf{e}}_{\mathbf{z}}$, we have

$$\begin{aligned}
\mathbf{r} &= R\,\hat{\mathbf{e}}_{\mathbf{R}} + z\,\hat{\mathbf{e}}_{\mathbf{z}} \\
\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \hat{\mathbf{e}}_{\mathbf{R}}\frac{\mathrm{d}R}{\mathrm{d}t} + \hat{\mathbf{e}}_{\boldsymbol{\phi}}\,R\frac{\mathrm{d}\phi}{\mathrm{d}t} + \hat{\mathbf{e}}_{\mathbf{z}}\frac{\mathrm{d}z}{\mathrm{d}t} \\
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \hat{\mathbf{e}}_{\mathbf{R}}\left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right) + \hat{\mathbf{e}}_{\boldsymbol{\phi}}\left(2\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\right) + \hat{\mathbf{e}}_{\mathbf{z}}\frac{\mathrm{d}^2z}{\mathrm{d}t^2}
\end{aligned}$$

for any position, velocity and acceleration.

C6. Some General Vector Identities

The product rule for differentiating the scalar product of two vectors is

$$\frac{\mathrm{d}(\mathbf{a}\,\boldsymbol{.}\,\mathbf{b})}{\mathrm{d}t} \equiv \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}\boldsymbol{.}\,\mathbf{b} + \mathbf{a}\,\boldsymbol{.}\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}\ ,$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$ that are functions of a scalar variable t.

C7. Gauss's Theorem (the Divergence Theorem)

Gauss's Theorem (the Divergence Theorem) states that

$$\int_V (\boldsymbol{\nabla}\,\boldsymbol{.}\,\mathbf{A})\,\mathrm{d}V \equiv \int_S \mathbf{A}\,\boldsymbol{.}\,\mathrm{d}\mathbf{S}$$

for any continuous vector field $\mathbf{A}$ over any volume V, where S is the surface that bounds the volume V.