Appendix B

Revision of Newtonian Gravitation

B1. Summary

This appendix summarises some basic results relating to gravitation from Newtonian Mechanics. This information covers most of the basic principles from physics about gravitation that are needed for the course.

B2. The Gravitational Field from a Point Mass

The attractive force between two particles of mass m_1 and m_2 a distance r apart is

$$F_{grav} = \frac{G m_1 m_2}{r^2} \ ,$$

where G is the universal constant of gravitation, with $G = 6.673 \times 10^{-11}$ m^3 kg^{-1}s^{-2}.
Using Newton's Second Law, the acceleration due to gravity at a distance r from a point mass m is

$$g = \frac{Gm}{r^2} \ ,$$

directed towards the point mass.
The acceleration due to gravity is the gravitational field strength.
The gravitational potential a distance r from a point mass m is

$$\Phi = -\frac{Gm}{r} \ .$$

In vector notation, the gravitational acceleration at a point with position vector $\mathbf{r}$ caused by a point mass m at position vector $\mathbf{r}_p$ is

$$\mathbf{g} = -\frac{Gm}{|\mathbf{r} - \mathbf{r}_p|^3} (\mathbf{r} - \mathbf{r}_p) \ .$$

The gravitational potential at position $\mathbf{r}$ is then

$$\Phi = -\frac{Gm}{|\mathbf{r} - \mathbf{r}_p|} \ .$$

B3. General Results about Gravitational Fields

The results in this section apply to any gravitational field, whether caused by point masses or continuous distributions of mass, and apply outside or inside any distribution of mass.

The acceleration due to gravity $\mathbf{g}$ at any point is related to the gradient of the gravitational potential Φ by

$$\mathbf{g} \;=\; -\,\boldsymbol{\nabla}\Phi \;\;,$$

in any gravitational field.
The potential is negative at all times, tending to zero at infinite distance. The S.I. unit of potential is $\mathrm{m^2\,s^{-2}}$.
The potential energy of a particle of mass m at a point in a gravitational field is

$$U \;=\; m\,\Phi \;\;,$$

where Φ is the potential at the point. This defintion means that the gravitational potential energy is negative.
Gauss's Law relates the integral of the gravitational acceleration over a closed surface to the mass lying inside that surface. If $\mathbf{g}$ is the acceleration due to gravity and $\mathrm{d}\mathbf{S}$ is an element of the surface S, then

$$\int_S \mathbf{g}\boldsymbol{.}\mathrm{d}\mathbf{S} \;=\; -\,4\pi\,G\,M_S$$

for any closed surface S, where M_S is the total mass contained within the surface. This is the direct equivalent of Gauss's Law for electrostatics ($\int_S \mathbf{E}\boldsymbol{.}\mathrm{d}\mathbf{S} = Q_S/\epsilon_0$).
Substituting for $\mathbf{g} = -\,\boldsymbol{\nabla}\Phi$, we also have

$$\int_S \boldsymbol{\nabla}\Phi\boldsymbol{.}\mathrm{d}\mathbf{S} \;=\; +\,4\pi\,G\,M_S$$

The Poisson Equation relates the Laplacian of the potential at a point to the mass density. It states that

$$\nabla^2\Phi \;=\; 4\pi\,G\,\rho \;\;,$$

where Φ is the potential at the point and ρ is the density.

B4. Distributions of point masses

In this section we shall consider the gravitational effects of a series of point masses m_i which are located at positions $\mathbf{r}_i$, for $i = 1, N$.
The gravitational potential at some position $\mathbf{r}$ caused by the distribution is

$$\Phi(\mathbf{r}) \;=\; -\,G\sum_{i=1}^{N}\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|} \quad ,$$

where G is the constant of gravitation.
The acceleration due to gravity at the point $\mathbf{r}$ is

$$\mathbf{g} \;=\; -\,G\sum_{i=1}^{N}\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|^3}\,(\mathbf{r}-\mathbf{r}_i) \quad .$$

The internal gravitational potential energy of the distribution of point masses is

$$U \;=\; -\,\tfrac{1}{2}\,G\sum_{\substack{i,j\\ i\neq j}}\frac{m_i\,m_j}{|\mathbf{r}_i-\mathbf{r}_j|} \quad .$$

B5. Continuous distributions of mass

The gravitational potential at a position $\mathbf{r}$ in a continuous distribution of mass enclosed in a volume V is given by

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}V' \quad .$$

where $\mathbf{r}'$ is the position vector of the volume element $\mathrm{d}V'$, $\rho(\mathbf{r}')$ is the mass density at the position $\mathbf{r}'$, and G is the constant of gravitation. The gradient of this is

$$\boldsymbol{\nabla}\Phi(\mathbf{r}) = G \int_V \frac{\rho(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \, \mathrm{d}V' \quad .$$

So the acceleration due to gravity at the point $\mathbf{r}$ is

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \, \mathrm{d}V' \quad .$$

The internal gravitational potential energy of some distribution of mass is

$$U = -\tfrac{1}{2}G \int_V \int_V \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}V \, \mathrm{d}V' \quad .$$

where $\mathbf{r}$ is the position vector of the volume element $\mathrm{d}V$ and where $\mathbf{r}'$ is the position vector of the volume element $\mathrm{d}V'$.

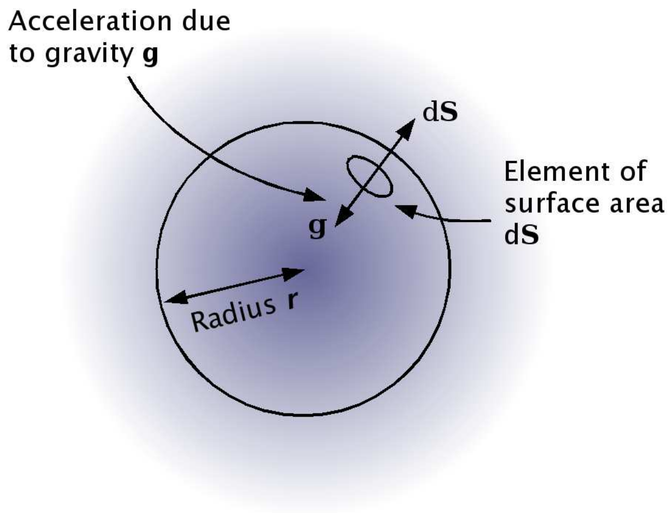
B6. The Gravitational Field within a Spherically Symmetric Distribution of Mass

The acceleration due to gravity at a distance r from the centre of a spherically symmetric distribution of mass is

$$g = \frac{G\,M(r)}{r^2}$$

where $M(r)$ is the mass interior to a radius r, and is directed towards the centre of the distribution. This result does not depend on how the mass is distributed, other than it is spherically symmetric. Mass outside the radius r does not affect the gravitational field at r in this spherically symmetric case. This is the same acceleration as would be given by a point mass $M(r)$ at the centre of the distribution.

This result can be derived very easily using Gauss's Law.



Consider a spherical surface S of radius r centred on the distribution.

The acceleration due to gravity at a point on the surface is $\mathbf{g}$. The magnitude of the acceleration everywhere on the surface is $|\mathbf{g}| \equiv g$, from symmetry.

From Gauss's Law,

$$\int_{\mathrm{S}} \mathbf{g} \cdot \mathrm{d}\mathbf{S} \quad = \quad -\, 4\pi G M(r) \quad ,$$

where $M(r)$ is the mass inside the surface and G is the universal constant of gravitation.
But an element of the surface area $\mathrm{d}\mathbf{S}$ is anti-parallel to the acceleration due to gravity $\mathbf{g}$, so $\mathbf{g} \cdot \mathrm{d}\mathbf{S} \; = \; -\,|\mathbf{g}|\,|\mathrm{d}\mathbf{S}| \; = \; -\,g\,\mathrm{d}S$. But since g is constant over the spherical surface,

$$\begin{aligned} -\, g \int_S \mathrm{d}S \quad &= \quad -\, 4\pi G M(r) \\ \therefore \qquad -\, g\,(4\pi r^2) \quad &= \quad -\, 4\pi G M(r) \quad , \end{aligned}$$

which gives,

$$g \quad = \quad \frac{G\,M(r)}{r^2} \quad .$$

This analysis is possible because of the spherical symmetry.
The gravitational potential Φ at a distance r from the centre of the spherically symmteric distribution is therefore

$$\Phi \quad = \quad \frac{G\,M(r)}{r} \quad .$$

The internal gravitational potential energy of some spherically symmetric distribution of mass can be obtained by considering the potential energy of a thin spherical shell within the distribution and integrating over all such shells. Consider a thin shell of radius r and thickness $\mathrm{d}r$ centred on the distribution. The gravitational potential at the shell is $\Phi = -GM(r)/r$, where $M(r)$ is the mass inside the shell, using the result above. The shell will have a mass $\mathrm{d}M = 4\pi r^2 \rho(r)\,\mathrm{d}r$, where $\rho(r)$ is the density. The potential energy of the shell due to the distribution of mass is

$$\mathrm{d}U \quad = \quad \Phi\,\mathrm{d}M \quad = \quad -\,\frac{GM(r)}{r}\,\mathrm{d}M \quad = \quad -\,4\pi G\,r\,M(r)\,\rho\,\mathrm{d}r \quad .$$

Integrating over the whole distribution (from $r = 0$ to infinity),

$$U \;=\; -\,G\int_0^{M_{tot}} \frac{M(r)\,\mathrm{d}M}{r} \;=\; -\,4\pi G\int_0^{\infty} r\,M(r)\,\rho(r)\,\mathrm{d}r$$

where M_{tot} is the total mass of the distribution. Either of these expressions for U can be used, depending on which is most convenient for the particular circumstances. For the case of a uniform sphere of mass M_{tot} and radius R, this gives

$$U \;=\; -\,\frac{3}{5}\,\frac{GM_{tot}^2}{R} \quad .$$

B8. Potentials Within Galaxies

It is very difficult to measure the gravitational potentials Φ of galaxies with accuracy because of the poor constraints on the dark matter far from their centres. However, the gradients $\boldsymbol{\nabla}\Phi$ of the potentials can be measured directly from the orbital velocities of stars and gas with some precision.