RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 9

1. To compute the timelike geodesic equations recall that

$$L = -\left(1 - \frac{2GM}{r}\right)\dot{t}^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2$$

The geodesic equations are then obtained by noticing that

$$\frac{\partial L}{\partial t} = 0, \quad \frac{\partial L}{\partial \varphi} = 0, \quad \frac{\partial L}{\partial r} = 2r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - A^{-2}A'\dot{r}^2 - A'\dot{t}^2, \quad \frac{\partial L}{\partial \theta} = r^2\sin\theta\cos\theta\dot{\varphi}^2,$$

where

$$A = 1 - \frac{2GM}{r}, \quad A' = \frac{2GM}{r^2},$$

and $'$ denotes the derivative with respect to r. The actual expressions are then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(A\dot{t}\right) = 0, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2A^{-1}\dot{r}\right) - 2r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + A^{-2}A'\dot{r}^2 + A'\dot{t}^2 = 0, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\theta}\right) - r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\dot{\varphi}\right) = 0. \tag{4}$$

In addition one has equation (5).

2. For a timelike geodesic one has that

$$\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\varphi}^2 = 1. \tag{5}$$

A radial geodesic is defined by the conditions

$$\dot{\theta} = \dot{\varphi} = 0.$$

Now, the geodesic equation for the time coordinate gives

$$\dot{t} = l\left(1 - \frac{2GM}{r}\right)^{-1}$$

Substituting this into equation (5) one obtains

$$l^2\left(1 - \frac{2GM}{r}\right)^{-1} - 1 = \dot{r}\left(1 - \frac{2GM}{r}\right)^{-1},$$

so that

$$\dot{r}^2 = l^2 - 1 + \frac{2GM}{r}.$$

If $l = 1$, the latter gives

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{\frac{2GM}{r}}.$$

Thus,

$$\int_0^{2GM} r^{1/2}\mathrm{d}r = \sqrt{2GM}\int_{\tau_1}^{\tau_2} = \sqrt{2GM}(\tau_2-\tau_1) = \sqrt{2GM}\Delta\tau,$$

from where

$$\frac{2}{3}(2GM)^{3/2} = (2GM)^{1/2}\Delta\tau.$$

Hence

$$\Delta\tau = \frac{4GM}{3}.$$

3. Now consider an orbit in the Equatorial plane ($\theta = \pi/2$). Hence $\dot{\theta} = 0$. Furthermore, the orbit is circular so that $r = D$, and $\dot{r} = 0$. The integration of equation (1) gives

$$\dot{t} = l\left(1-\frac{2GM}{r}\right)^{-1} = l\left(1-\frac{2GM}{D}\right)^{-1}, \quad l \text{ a constant.}$$

Also, from equation 4 one has

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\varphi}\right) = 0 \Rightarrow r^2\dot{\varphi} = h, \qquad \text{a constant.}$$

Notice that equation (2) reduces to

$$-2r\dot{\varphi}^2 + A'\dot{t}^2 = 0$$

Thus, substituting the previous two equations one has that

$$-2\frac{h^2}{D^3} + l^2\left(1-\frac{2GM}{D}\right)^{-2}\frac{2GM}{D^2} = 0 \Rightarrow l^2 = \left(1-\frac{2GM}{D}\right)^2\frac{h^2}{GMD}. \tag{6}$$

Now, for this problem, equation (5) gives

$$1 = \left(1-\frac{2GM}{D}\right)\dot{t}^2 - D^2\dot{\varphi}^2 = 1 = l^2\left(1-\frac{2GM}{D}\right)^{-1} - \frac{h^2}{D^2}.$$

Substituting (6) into the last equation one obtains

$$\left(1-\frac{2GM}{D}\right)\frac{h^2}{GMD} - \frac{h^2}{D^2} = 1 \Rightarrow h^2 = GMD\left(1-\frac{3GM}{D}\right)^{-1}.$$

Finally, following the hint, the time elapsed in one orbit is given by

$$\tau = \int_0^{2\pi}\frac{\mathrm{d}\tau}{\mathrm{d}\varphi}\mathrm{d}\varphi = \int_0^{2\pi}\dot{\varphi}^{-1}\mathrm{d}\varphi = \int_0^{2\pi}\frac{D^2}{h}\mathrm{d}\varphi = \frac{2\pi D^2}{h},$$

where equation (4) has been used. Substituting the obtained value for h one gets

$$\tau = 2\pi\left(\frac{D^3}{GM}\right)^{1/2}\left(1-\frac{3GM}{D}\right)^{1/2},$$

as required.