RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 8

1. In this case one has that
$$L = -e^{2Ar}\dot{t}^2 + \dot{r}^2.$$
Now, for the t components one has that
$$\frac{\partial L}{\partial t} = 0, \quad \frac{\partial L}{\partial \dot{t}} = -2e^{2Ar}\dot{t}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(-2e^{2Ar}\dot{t}\right) = \ddot{t}e^{2Ar} + 2Ae^{2Ar}\dot{r}\dot{t} = 0.$$
Thus, the Euler-Lagrange equation is given by
$$\ddot{t} + 2A\dot{r}\dot{t} = 0.$$
Now, comparing with th geodesic equation
$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = \ddot{t} + \Gamma^t{}_{tt}\dot{t}^2 + 2\Gamma^t{}_{tr}\dot{t}\dot{r} + \Gamma^t{}_{rr}\dot{r}^2 = 0.$$
Comparing one gets
$$\Gamma^t_{tt} = 0, \quad \Gamma^t{}_{tr} = \Gamma^t{}_{rt} = A, \quad \Gamma^t{}_{rr} = 0.$$
For the r components one has that
$$\frac{\partial L}{\partial r} = -2Ae^{2Ar}\dot{t}^2, \quad \frac{\partial L}{\partial \dot{r}} = 2\dot{r}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) = 2\ddot{r}.$$
Hence, the Euler-Lagrange equation is given by
$$\ddot{r} + Ae^{2Ar}\dot{t}^2 = 0.$$
Again, comparison with the geodesic equation gives
$$\Gamma^r{}_{rr} = 0, \quad \Gamma^r{}_{rt} = \Gamma^r{}_{tr} = 0. \quad \Gamma^r{}_{tt} = Ae^{2Ar}.$$
Using the formula for the Ricci tensor one has that
$$\begin{aligned}
R_{tt} = R_{00} &= \partial_a\Gamma^a{}_{00} - \partial_0\Gamma^a{}_{0a} + \Gamma^a{}_{ea}\Gamma^e{}_{00} - \Gamma^a{}_{e0}\Gamma^e{}_{0a}, \\
&= \partial_0\Gamma^0{}_{00} + \partial_1\Gamma^1{}_{00} - \partial_0\Gamma^0{}_{00} - \partial_0\Gamma^1{}_{01} + \Gamma^0{}_{e0}\Gamma^e{}_{00} + \Gamma^1{}_{e1}\Gamma^e{}_{00} - \Gamma^0{}_{e0}\Gamma^e{}_{00} - \Gamma^1{}_{e0}\Gamma^e{}_{01}, \\
&= \partial_1\Gamma^1{}_{00} + \Gamma^0{}_{e0}\Gamma^e{}_{00} + \Gamma^1{}_{e1}\Gamma^e{}_{00} - \Gamma^0{}_{e0}\Gamma^e{}_{00} - \Gamma^1{}_{e0}\Gamma^e{}_{01}, \\
&= \partial_1\Gamma^1{}_{00} + \Gamma^0{}_{00}\Gamma^0{}_{00} + \Gamma^0{}_{10}\Gamma^1{}_{00} + \Gamma^1{}_{01}\Gamma^0{}_{00} + \Gamma^1{}_{11}\Gamma^1{}_{00} - \Gamma^0{}_{00}\Gamma^0{}_{00} - \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01} \\
&\quad -\Gamma^1{}_{10}\Gamma^1_{01}, \\
&= \partial_1\Gamma^1{}_{00} + \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^0{}_{10}\Gamma^1{}_{00} - \Gamma^1{}_{00}\Gamma^0{}_{01}, \\
&= A\partial_r\left(e^{2ar}\right) - A^2e^{2Ar} = A^2e^{2Ar}.
\end{aligned}$$

2. (i) Starting from the definition of the Christoffel symbols:
$$\begin{aligned}
\Gamma_{abc} &\equiv g_{af}\Gamma^f_{bc} \\
&= g_{af}\Gamma^f_{bc} = \frac{1}{2}g_{af}g^{fe}(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \\
&= \frac{1}{2}\delta_a{}^e(\partial_b g_{ec} + \partial_c g_{be} - \partial_e g_{bc}) \\
&= \frac{1}{2}(\partial_b(\delta_a{}^e g_{ec}) + \partial_c(\delta_a{}^e g_{be}) - \partial_a g_{bc}) \\
&= \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}).
\end{aligned}$$

(ii) One can verify this by direct inspection. For example,

$$R_{bacd} = K(g_{bc}g_{ad} - g_{bd}gca) = -K(g_{ac}g_{bd} - g_{ad}g_{cb}) = -R_{abcd},$$

where it has been used that $g_{ab} = g_{ba}$. To compute $\nabla_e R_{abcd}$ one uses the Leibnitz rule:

$$\nabla_e R_{abcd} = K((\nabla_e g_{ac})g_{bd} + g_{ac}(\nabla_e g_{bd}) - (\nabla_e g_{ac})g_{bd} - g_{ac}(\nabla_e g_{bd})) = 0.$$

To compute R_{bd} proceed as folows:

$$\begin{aligned} R_{bd} &= g^{ac}R_{abcd} = Kg^{ac}(g_{ac}g_{bd} - g_{ad}g_{cb}) = K(g^{ac}g_{ac}g_{bd} - g^{ac}g_{ad}g_{cb}) \\ &= K(\delta_a{}^a g_{bd} - \delta_d{}^c g_{cb}) \\ &= K(4g_{bd} - g_{bd}) = 3Kg_{bd}, \end{aligned}$$

where it has been used that $\delta_a{}^a = 4$ —see Coursework 6. Finally

$$R = g^{bd}R_{bd} = 3Kg^{bd}g_{bd} = 12K.$$

3. (i) Start from the expression for the Riemann tensor in locally inertial coordinates:

$$R_{abcd} = \frac{1}{2}(\partial_d\partial_a g_{bc} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd} - \partial_d\partial_b g_{ac}).$$

Now, performing the substitutions $b \to c \to d \to b$ twice one obtains

$$\begin{aligned} R_{acdb} &= \frac{1}{2}(\partial_b\partial_a g_{cb} + \partial_d\partial_c g_{ab} - \partial_d\partial_a g_{cb} - \partial_b\partial_c g_{ad}), \\ R_{adbc} &= \frac{1}{2}(\partial_c\partial_a g_{dc} + \partial_b\partial_d g_{ac} - \partial_b\partial_a g_{dc} - \partial_c\partial_d g_{ab}). \end{aligned}$$

Now, adding and recalling that partial derivatives commute and that g_{ab} is symmetric one obtains the desired result.

(ii) There is a typo in this expression and it should read

$$R_{a[bcd]} = 0.$$

Note that (see notes on symmetric and atisymmetric parts of a tensor):

$$R_{a[bcd]} = \frac{1}{6}\left(R_{abcd} + R_{acdb} + R_{adbc} - R_{acbd} - R_{adcb} - R_{abdc}\right).$$

Recall, however, that the Riemann tensor is antisymmetric under the interchange of the last two indices —e.g. $R_{abcd} = -R_{abdc}$. Thus,

$$R_{a[bcd]} = \frac{1}{3}\left(R_{abcd} + R_{acdb} + R_{adbc}\right) = 0.$$

Also notice that the wrong expression

$$R_{a(bcd)} = 0$$

is also true. This again follows from the antisymmetry of the Riemann tensor on the last two indices:

$$R_{abcd} = -R_{abdc} \Rightarrow R_{a(bcd)} = -R_{a(bdc)} = -R_{a(bcd)},$$

from where it follows that

$$R_{a(bcd)} = 0.$$