RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 7

1. Taking differentials from the coordinate transformation

$$\begin{aligned}
\mathrm{d}x &= \mathrm{d}r\sin\theta\sin\varphi + r\cos\theta\sin\varphi\mathrm{d}\theta + r\sin\theta\cos\varphi\mathrm{d}\varphi,\\
\mathrm{d}y &= \mathrm{d}r\sin\theta\cos\varphi + r\cos\theta\cos\varphi\mathrm{d}\theta - r\cos\theta\sin\varphi\mathrm{d}\varphi,\\
\mathrm{d}z &= \mathrm{d}r\cos\theta - r\sin\theta\mathrm{d}\theta.
\end{aligned}$$

One then takes squares from this expression and substitutes in the line element. Using the standard identities

$$\sin^2\theta + \cos^2\theta = 1, \quad \sin^2\varphi + \cos^2\varphi = 1,$$

one obtains the desired result —in particular notice that all cross terms like $\mathrm{d}r\mathrm{d}\theta$, etc. cancel out.

2. Dividing the line element by $\mathrm{d}\lambda^2$ one obtains

$$L = \left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = -y^3\dot{x}^2 + x^4\dot{y}^2.$$

For $i = 1$ the Euler-Lagrange equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = 0.$$

A calculation shows that

$$\frac{\partial L}{\partial x} = 4x^3\dot{y}^2, \quad \frac{\partial L}{\partial\dot{x}} = -2y^3\dot{x}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\dot{x}}\right) = -6y^2\dot{y}\dot{x} - 2y^3\ddot{x}.$$

Putting all together and simplifying one gets

$$\ddot{x} + \frac{3}{y}\dot{x}\dot{y} + 2\frac{x^3}{y^3}\dot{y} = 0.$$

From here one can directly read

$$\Gamma^1_{11} = 0, \quad \Gamma^1_{12} = \frac{3}{2y}, \quad \Gamma^1_{22} = \frac{2x^3}{y^3}.$$

For $i = 2$ the Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\dot{y}}\right) - \frac{\partial L}{\partial y} = 0.$$

Again, a computation yields

$$\frac{\partial L}{\partial y} = -3y^2\dot{x}^2, \quad \frac{\partial L}{\partial\dot{y}} = 2x^4\dot{y}, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial\dot{y}}\right) = 8x^3\dot{x}\dot{y} + 2x^4\ddot{y}.$$

It follows then that

$$\ddot{y} + \frac{4}{x}\dot{x}\dot{y} + \frac{3y^2}{2x^4}\dot{x}^2 = 0,$$

from where one reads

$$\Gamma^2_{22} = 0, \quad \Gamma^2_{11} = \frac{3y^2}{2x^4}, \quad \Gamma^2_{12} = \frac{2}{x}.$$

Putting this information into the formula for the Riemann tensor one has

$$\begin{aligned}
R^1{}_{212} &= \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1_{e1}\Gamma^e_{22} - \Gamma^1_{e2}\Gamma^e_{21}, \\
&= \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} + \Gamma^1{}_{11}\Gamma^1_{22} + \Gamma^1_{21}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^1_{22}\Gamma^2_{21} \\
&= \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^1_{22}\Gamma^2_{21} \\
&= \frac{2x^2}{y^3} - \frac{3}{4y^2}.
\end{aligned}$$

3. By definition

$$\begin{aligned}
T_{(ab)} &= \frac{1}{2}\left(T_{ab} + T_{ba}\right), \\
T_{[ab]} &= \frac{1}{2}\left(T_{ab} - T_{ba}\right).
\end{aligned}$$

Adding one finds that

$$T_{ab} = T_{(ab)} + T_{[ab]}.$$

Now,

$$g^{ab}T_{ab} = g^{ab}T_{(ab)} + g^{ab}T_{[ab]}.$$

Recalling that g^{ab} is symmetric one obtains (following the same argument as in question 5 of CW5 that

$$g^{ab}T_{[ab]} = -g^{ab}T_{[ba]} = -g^{ba}T_{[ba]} = -g^{ab}T_{[ab]},$$

from where it follows that

$$g^{ab}T_{[ab]} = 0.$$

4. A local inertial frame at a point p are coordinates such that at the point p the metric is the Minkowski metric, the first derivatives of the metric vanish, but the second derivatives do not. Hence the Christoffel symbols vanish at p, but not their derivatives. Important: these properties are only valid at p!

The expression for the Riemann tensor follows from the notes. Now, interchanging a and b in the expression one obtains

$$R_{bacd} = \frac{1}{2}(\partial_d\partial_b g_{ac} + \partial_c\partial_a g_{bd} - \partial_c\partial_b g_{ad} - \partial_d\partial_a g_{bc}) = -R_{abcd}$$

as required.

See the definitions for the Ricci tensor and scalar in the notes. To prove that R_{bd} is symmetric note that

$$R_{bd} = R^a{}_{bad} = g^{ae}R_{ebad} = g^{ae}R_{adeb} = g^{ea}R_{adeb} = R^e{}_{deb} = R_{db}.$$

In this chain of equations it has been used that g^{ae} is symmetric and that $R_{ebad} = R_{adeb}$. Notice that one can only use the symmetries of the tensor if all the indices are in the same position —that is, all covariant or all contravariant.