RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 6

1.
(i) The required formula is a combination of the formulae for the covariant derivative of $(0,1)$ and $(1,0)$ tensors:

$$\nabla_a S_b{}^c = \partial_a S_b{}^c - \Gamma^e_{ba} S_e{}^c + \Gamma_{ea}{}^c S_b{}^e.$$

(ii) To compute the covariant derivative of the Kronecker delta one uses the formula obtained in (i). Setting $S_b{}^c = \delta_b{}^c$ one has

$$\nabla_a \delta_b{}^c = \partial_a \delta_b{}^c - \Gamma^e_{ba} \delta_e{}^c + \Gamma_{ea}{}^c \delta_b{}^e.$$

Now, $\delta_b{}^c$ takes the values 0 or 1. Hence $\partial_a \delta_b{}^c = 0$. Thus,

$$\nabla_a \delta_b{}^c = -\Gamma^e_{ba} \delta_e{}^c + \Gamma_{ea}{}^c \delta_b{}^e = -\Gamma^c_{ba} + \Gamma^c_{ba},$$

where in the second equality one uses $W_e \delta_a{}^e = \delta_a$ and $V^e \delta_e{}^b = V^b$ for arbitrary W_a and V^b. Accordingly,

$$\nabla_a \delta_b{}^c = 0$$

as required.
(iii) One has that

$$\delta_a{}^a = \sum_{a=0}^{3} \delta_a{}^a = \sum_{a=0}^{3} 1 = 4.$$

2. The tensor $R^a{}_{bcd}$ is of type $(1,3)$. Thus, its transformation law is given by

$$R'^a{}_{bcd} = \frac{\partial x'^a}{\partial x^e} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^c} \frac{\partial x^h}{\partial x'^d} R^e{}_{fgh}$$

Contracting a and c one gets

$$\begin{aligned}
R'^a{}_{bad} &= \frac{\partial x'^a}{\partial x^e} \frac{\partial x^f}{\partial x'^b} \frac{\partial x^g}{\partial x'^a} \frac{\partial x^h}{\partial x'^d} R^e{}_{fgh}, \\
&= \left(\frac{\partial x'^a}{\partial x^e} \frac{\partial x^g}{\partial x'^a} \right) \frac{\partial x^f}{\partial x'^b} \frac{\partial x^h}{\partial x'^d} R^e{}_{fgh}, \\
&= \left(\frac{\partial x^g}{\partial x^e} \right) \frac{\partial x^f}{\partial x'^b} \frac{\partial x^h}{\partial x'^d} R^e{}_{fgh}, \\
&= \delta_e{}^g \frac{\partial x^f}{\partial x'^b} \frac{\partial x^h}{\partial x'^d} R^e{}_{fgh}, \\
&= \frac{\partial x^f}{\partial x'^b} \frac{\partial x^h}{\partial x'^d} R^e{}_{feh},
\end{aligned}$$

as required. Note that to pass from the second to the third line the chain rule has been used.
3. Following the hint one has that

$$\nabla_b (V_a W^a) = \nabla_b V_a W^a + V_a \nabla_b W^a,$$

and also that

$$\nabla_b(V_aW^a) = \partial_b(V_aW^a) + \partial_b V_a W^a + V_a\partial_b W^a.$$

From the definition of covariant derivative for contravariant tensors one has that

$$\nabla_b W^a = \partial_b W^a + \Gamma^a{}_{cb}W^c,$$

so that

$$\partial_b W^a = \nabla_b W^a - \Gamma^a{}_{cb}W^c.$$

Thus, substituting in the above expressions renders

$$\nabla_b V_a W^a + V_a\nabla_b W^a = \partial_b(V_aW^a) = \partial_b V_a W^a + V_a\left(\nabla_b W^a - \Gamma^a{}_{cb}W^c\right),$$

which simplifies to

$$\nabla_b V_a W^a + V_a\nabla_b W^a = \partial_b(V_aW^a) = \partial_b V_a W^a + V_a\left(\nabla_b W^a - \Gamma^a{}_{cb}W^c\right),$$

and in turn to

$$\nabla_b V_a W^a = \partial_b V_a W^a - \Gamma^a{}_{cb}W^c V_a.$$

However, W^a is arbitrary, so the desired result follows.
4. One directly sees that

$$(g_{ab}) = \begin{pmatrix} e^y & 0 \\ 0 & e^x \end{pmatrix}, \quad (g^{ab}) = \begin{pmatrix} e^{-y} & 0 \\ 0 & e^{-x} \end{pmatrix}.$$

Now, recalling the identification $(x^1, x^2) = (x, y)$ one has that

$$\begin{aligned}\Gamma^1{}_{11} &= \frac{1}{2}g^{1e}(\partial_1 g_{e1} + \partial_1 g_{1e} - \partial_e g_{11}, \\ &= \frac{1}{2}g^{11}(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = 0,\end{aligned}$$

as g_{11} depends only on $x^2 = y$. One also has that

$$\begin{aligned}\Gamma^1{}_{12} &= \frac{1}{2}g^{1e}(\partial_1 g_{e2} + \partial_2 g_{1e} - \partial_e g_{12}, \\ &= \frac{1}{2}g^{11}(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}), \\ &= \frac{1}{2}g^{11}\partial_2 g_{11} = \frac{1}{2}e^{-y}e^y = \frac{1}{2}.\end{aligned}$$

By symmetry one has that

$$\Gamma^1{}_{12} = \Gamma^1{}_{21} = \frac{1}{2}.$$

Now for

$$\begin{aligned}\Gamma^2{}_{11} &= \frac{1}{2}g^{2e}(\partial_1 g_{e1} + \partial_1 g_{1e} - \partial_e g_{11}), \\ &= \frac{1}{2}g^{22}(\partial_1 g_{21} + \partial_1 g_{12} - \partial_2 g_{11}), \\ &= -\frac{1}{2}g^{22}\partial_2 g_{11} = -\frac{1}{2}e^{x-y}.\end{aligned}$$