RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 5

1.
(i) The transformation laws are, respectively

$$V'_i = \frac{\partial x^a}{\partial x'^i} V_a, \quad W'^{jk}{}_l = \frac{\partial x'^j}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} W^{bc}{}_d.$$

(ii) Let $Y_{il}{}^{jk} \equiv V_i W^{jk}{}_l$. Using the result from (i) one has that

$$\begin{aligned} Y'^{jk}_{il} &= V'_i W'^{jk}{}_l \\ &= \frac{\partial x^a}{\partial x'^i} V_a \frac{\partial x'^j}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} W^{bc}{}_d \\ &= \frac{\partial x^a}{\partial x'^i} \frac{\partial x'^j}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} V_a W^{bc}{}_d \\ &= \frac{\partial x^a}{\partial x'^i} \frac{\partial x'^j}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{bc}, \end{aligned}$$

which is the transformation rule for a $(2,2)$ tensor.
(iii) Let $Y_{il}{}^{jk}$ be a tensor of type $(2,2)$. From (ii) one has that its transformation rule is

$$Y'^{jk}_{il} = \frac{\partial x^a}{\partial x'^i} \frac{\partial x'^j}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{bc}.$$

Contract i and j:

$$\begin{aligned} Y'^{ik}_{il} &= \frac{\partial x^a}{\partial x'^i} \frac{\partial x'^i}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{bc}, \\ &= \frac{\partial x^a}{\partial x^b} \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{bc} \\ &= \delta^a{}_b \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{bc} \\ &= \frac{\partial x'^k}{\partial x^c} \frac{\partial x^d}{\partial x'^l} Y_{ad}{}^{ac} \end{aligned}$$

which is the transformation rule of a $(1,1)$ tensor.

2. The transformation rules for A_i and B^i are

$$A'_i = \frac{\partial x^k}{\partial x'^i} A_k, \quad B'^i = \frac{\partial x'^i}{\partial x_l} B^l.$$

Now,

$$\begin{aligned} A'_i B'^i &= \frac{\partial x^k}{\partial x'^i} A_k \frac{\partial x'^i}{\partial x_l} B^l \\ &= \frac{\partial x^k}{\partial x'^i} A_k \frac{\partial x'^i}{\partial x_l} A_k B^l, \\ &= \frac{\partial x^k}{\partial x^l} A_k B^l \\ &= \delta^k{}_l A_k B^l \\ &= A_k B^k, \end{aligned}$$

as required.

3. As $T^{ab}V_aW_b$ is a scalar one has that

$$T^{ab}V_aW_b = T'^{ab}V'_aW'_b.$$

Now, V_a and W_b are covariant so that

$$T^{ab}V_aW_b = T'^{ab}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}V_eW_f.$$

From here it follows that

$$\left(T^{ef} - T'^{ab}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}\right)V_eW_f = 0,$$

where it has been used that $T^{ab}V_aW_b = T^{ef}V_eW_f$ as the indices a and b are dummy. As V_e and W_f are arbitrary, it follows that

$$T^{ef} - T'^{ab}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b} = 0 \Longrightarrow T^{ef} = T'^{ab}\frac{\partial x^e}{\partial x'^a}\frac{\partial x^f}{\partial x'^b}.$$

5. One uses the properties of the tensors as follows

$$A_{ij}B^{ij} = -A_{ij}B^{ji} = -A_{ji}B^{ji} = -A_{ij}B^{ij}.$$

The first equality follows from the antisymmetry of B^{ij}. The second from the symmetry of A_{ij}. The third from the fact that i and j are dummies so one can interchange them at the same time. It follows that $A_{ij}B^{ij} = 0$.

6. If V_{ab} is a tensor of type $(0,2)$, then

$$V'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}V_{cd}.$$

Now, because of $V_{ab} = V_{ba}$, one has that

$$V'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}V_{dc} = V'_{ba}.$$

Thus, the symmetry is valid in any coordinate system.