RELATIVITY (MTH6132)

SOLUTIONS TO THE PROBLEM SET 3

1. Probably the simplest way to solve the problem is to make use of the invariance of the interval:

$$-c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta t')^2 + (\Delta x')^2.$$

In the problem one has that $\Delta t = 0$ (events simultaneous in F), and that $\Delta x = 3m$. Also $\Delta t' = 10^{-8}s$. One can readily solve for $\Delta x'$, to obtain:

$$\Delta x' = 3\sqrt{2}m.$$

2. If $\bar{A}$ is a unit spacelike vector, then one has that $|\bar{A}| = 1$. From here it follows that

$$-(A^0)^2 + 4 = 1,$$

and finally that

$$A^0 = \pm\sqrt{3}.$$

If $\bar{A}$ and $\bar{B}$ are orthogonal (i.e. $\bar{A}\cdot\bar{B}$) then

$$-3A^0 + 2B^2 = 0,$$

from where

$$B^2 = \frac{3}{2}A^0 = \pm\frac{3\sqrt{3}}{2}.$$

3. If $\bar{A}$ and $\bar{B}$ are timelike then

$$|\bar{A}|^2 < 0, \quad |\bar{B}|^2 < 0.$$

Hence,

$$-(A^0)^2 + (A^1)^2 < 0, \quad -(B^0)^2 + (B^1)^2 < 0,$$

and furthermore

$$A^1 < A^0, \quad B^1 < B^0$$

since A^0, A^1, B^0 and B^1 are both positive. Adding one finds that

$$(A^1 + B^1)^2 < (A^0 + B^0)^2.$$

Finally, notice that

$$|\bar{A} + \bar{B}|^2 = -(A^0 + B^0)^2 + (A^1 + B^1)^2$$

which as a consequence of the previous inequality can never be zero.

4. Since $\bar{A}$ is a 4-vector, its components transform like (t, x, y, z) under Lorentz transformations. Set $c = 1$ for simplicity:

$$A'^0 = \gamma(A^0 - vA^1), \quad A'^1 = \gamma(A^1 - vA^0), \quad A'^2 = A^2, \quad A'^3 = A^3.$$

Now,

$$|\bar{A}'|^2 = -(A'^0)^2 + (A'^1)^2 + (A'^2)^2 + (A'^3)^2).$$

A direct substitution renderes then

$$\begin{aligned}|\bar{A}'|^2 &= \gamma^2(A^1)^2 + \gamma^2 v^2 (A^0)^2 - 2\gamma v A^1 A^0 - \gamma^2 (A^0)^2 - \gamma^2 v^2 (A^1)^2 + 2\gamma v A^0 A^1 + (A^2)^2 + (A^3)^2, \\ &= \gamma^2 (A^1)^2 (1 - v^2) - \gamma^2 (A^0)^2 (1 - v^2) + (A^2)^2 + (A^3)^2.\end{aligned}$$

However,

$$\gamma^2 = \frac{1}{1 - v^2},$$

so that one obtains the desired result. For the second part note that $\mathrm{d}t$, $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ transform like (t, x, y, z). Hence, $(\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ is an example of $\bar{A}$.

5. Crucial here is to remember that

$$\bar{U} = \gamma(1, \underline{v}),$$

and that

$$\gamma = (1 - v^2)^{-1/2} \equiv U^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau}.$$

Also that one write as well

$$\bar{U} = (U^0, U^1, U^2, U^3) = (U^0, U^\alpha).$$

From these one finds

1)

$$U^0 = \left(1 - v^2\right)^{-1/2}.$$

2)

$$U^\alpha = (1 - v^2)^{-1/2} v^\alpha.$$

3) Recall that $|\bar{U}|^2 = -1$, so that

$$U^0 = \sqrt{1 + (U^1)^2 + (U^2)^2 + (U^3)^2}.$$

4)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t} = (1 - v^2)^{-1/2} \frac{\mathrm{d}}{\mathrm{d}t}.$$

5)

$$v^\alpha = U^\alpha / U^0 = \frac{U^\alpha}{\sqrt{1 + (U^1)^2 + (U^2)^2 + (U^3)^2}}.$$

6) Using 1) one finds

$$|\underline{v}| = \sqrt{1 - (U^0)^{-2}}.$$