SOLAR SYSTEM

EXERCISE SHEET 3

SOLUTIONS

$\boxed{Q3.1}$ $h = na^2\sqrt{1-e^2}$

But $h = r^2\dot{\theta}$ where $\dot{\theta}$ is angular velocity
(N.B. $n = \dot{\theta}$ in circular case)

Also $$r = \frac{a(1-e^2)}{1+e\cos f}$$ (5)

i.e. $$r^2 = \frac{a^2(1-e^2)^2}{(1+e\cos f)^2} \;;\quad \dot{\theta} = \frac{h}{r^2} = \frac{na^2\sqrt{1-e^2}(1+e\cos f)^2}{a^2(1-e^2)^2}$$

i.e. $$\dot{\theta} = \frac{n\sqrt{1-e^2}}{(1-e^2)^2}(1+e\cos f)^2$$ (10)

at aphelion, (i.e. $f = \pi$) $$\dot{\theta} = \frac{n\sqrt{1-e^2}}{(1-e^2)^2}(1-e)^2$$

$$= \frac{n\sqrt{1-e^2}\,(1-e)^2}{(1+e)^2(1-e)^2} = \frac{n\sqrt{1-e^2}}{(1+e)^2}$$ (10)

Jupiter's mean motion, $n' = \dot{\theta}'$ (a constant, as it is a circular orbit).

at the 2:1 resonance, $n = 2n'$ (5)

Matching the angular velocity at aphelion gives

$$\dot{\theta} = \frac{n\sqrt{1-e^2}}{(1+e)^2} = n' = \frac{n}{2}$$

i.e. asteroid would be instantaneously stationary when e satisfies the equation

$$\frac{\sqrt{1-e^2}}{(1+e)^2} = \frac{1}{2}$$ (5)

i.e. $4(1-e^2) = (1+e)^4$

i.e. $4(1+e)(1-e) = (1+e)^4$

i.e. $4(1-e) = (1+e)^3$

or $4 - 4e = 1 + 3e + 3e^2 + e^3$ (10)

i.e. $e^3 + 3e^2 + 7e - 3 = 0$

The solutions are : $-1.68233 \pm 2.32308\,i$

or 0.364656

e must be real (and +ve), so answer is

$$e = 0.364656$$ (5)

$\boxed{Q3.2}$

$$\dot{\varpi} = \frac{1}{na^2 e}\frac{\partial R}{\partial e}$$

$$R = \frac{Gm'}{a'}\left\{\frac{A}{8}(e^2+e'^2) + \frac{B}{4}ee'\cos(\varpi'-\varpi)\right\}$$

i.e. $$\dot{\varpi} = \frac{Gm'}{a'}\cdot\frac{1}{na^2 e}\left\{\frac{A}{8}.2e + \frac{B}{4}e'\cos(\varpi'-\varpi)\right\}$$ (10)

Use $GM \simeq n^2 a^3$ ~~which~~ gives

$$n \simeq \frac{360^\circ}{a^{3/2}} = \frac{360}{(2.86)^{3/2}} = 74.43\ ^\circ/\text{y} = 7443\ ^\circ/\text{cent.}$$ (10)

and $$\dot{\varpi} = \frac{n^2 a^3}{a'}\frac{m'}{M}\cdot\frac{1}{na^2}\left\{\frac{A}{4} + \frac{B}{4}\frac{e'}{e}\cos(\varpi'-\varpi)\right\}$$

$$= \frac{n}{4}\frac{a}{a'}\frac{m'}{M}\left\{A + B\frac{e'}{e}\cos(\varpi'-\varpi)\right\}$$

$$= \frac{7443}{4}\cdot\frac{2.86}{5.20}\cdot 10^{-3}\left\{1.79185 + \frac{0.048}{0.15}(-1.18029) \times\cos(\varpi'-\varpi)\right\}$$

$$= 1.0234\left\{1.79185 - 0.37769\cos(\varpi'-\varpi)\right\}$$ (10)

i.e. $$(\dot{\varpi})_{\varpi'=\varpi} = 1.44725\ ^\circ/\text{cent}$$ (5)

$$(\dot{\varpi})_{\varpi'=\varpi+180^\circ} = 2.22031\ ^\circ/\text{cent.}$$ (5)

Therefore the precession rate is slower when the perihelia are aligned. Therefore, at any given time, there should be more asteroids with perihelia aligned with Jupiter.

10