SOLAR SYSTEM

EXERCISE SHEET 1

SOLUTIONS

Q 1.1 $\quad i_1, i_2 \in \{1, \ldots, i_{MAX}\} \qquad i_1 < i_2$

(a)

$i_{MAX} = 2 \qquad i_1, i_2 \in \{1, 2\}$

$$\frac{i_1}{i_2} = \frac{1}{2} \qquad N_r = 1$$ 1

$i_{MAX} = 3 \qquad i_1, i_2 \in \{1, 2, 3\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{1}{3} \qquad N_r = 3$$ 1

$i_{MAX} = 4 \qquad i_1, i_2 \in \{1, 2, 3, 4\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{3}, \frac{1}{4} \qquad N_r = 5$$ 1

$i_{MAX} = 5 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5},$$
$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5} \qquad N_r = 9$$ 1

$i_{MAX} = 6 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5, 6\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$
$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{2}{5}, \frac{3}{5} \qquad N_r = 11$$ 1

$i_{MAX} = 7 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5, 6, 7\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7},$$

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{2}{5},$$

$$\frac{2}{7}, \frac{3}{5}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7} \qquad N_r = 17 \qquad (1)$$

$i_{MAX} = 8 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5, 6, 7, 8\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8},$$

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{2}{5},$$

$$\frac{2}{7}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{7}, \frac{5}{7}, \frac{5}{8} \qquad N_r = 21 \qquad (1)$$

$i_{MAX} = 9 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9},$$

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{2}{5},$$

$$\frac{2}{7}, \frac{2}{9}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{7}, \frac{4}{9}, \frac{5}{7},$$

$$\frac{5}{8}, \frac{5}{9}, \frac{7}{9} \qquad N_r = 27 \qquad (1)$$

$i_{MAX} = 10 \qquad i_1, i_2 \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$$\frac{i_1}{i_2} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10},$$

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{2}{5},$$

$$\frac{2}{7}, \frac{2}{9}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{3}{10}, \frac{4}{7}, \frac{4}{9}, \frac{5}{7},$$

$$\frac{5}{8}, \frac{5}{9}, \frac{7}{9}, \frac{7}{10} \qquad N_r = 31 \qquad (2)$$

(b) $\varepsilon_{MAX} = \frac{1}{2}$ separation of two closest rationals

e.g. $i_{MAX} = 7$; the ordering is

$$\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3},$$
$$\frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$$

$$\frac{1}{6} - \frac{1}{7} = \frac{7-6}{42} = \frac{1}{42} \quad ; \quad \frac{6}{7} - \frac{5}{6} = \frac{36-35}{42} = \frac{1}{42}$$

i.e. the separation of two closest rationals is always

$$\frac{1}{i_{MAX}-1} - \frac{1}{i_{MAX}} \quad \underline{\text{or}} \quad \frac{i_{MAX}-1}{i_{MAX}} - \frac{i_{MAX}-2}{i_{MAX}-1}$$

$$\text{i.e. } 2\,\varepsilon_{MAX} = \frac{i_{MAX}-1}{i_{MAX}} - \frac{i_{MAX}-2}{i_{MAX}-1}$$

$$\varepsilon_{MAX} = \frac{1}{2}\left(\frac{i_{MAX}-1}{i_{MAX}} - \frac{i_{MAX}-2}{i_{MAX}-1}\right) \qquad (10)$$

alternatively,

$$\varepsilon_{MAX} = \frac{1}{2}\left(\frac{1}{i_{MAX}-1} - \frac{1}{i_{MAX}}\right)$$

(Check these are equivalent:

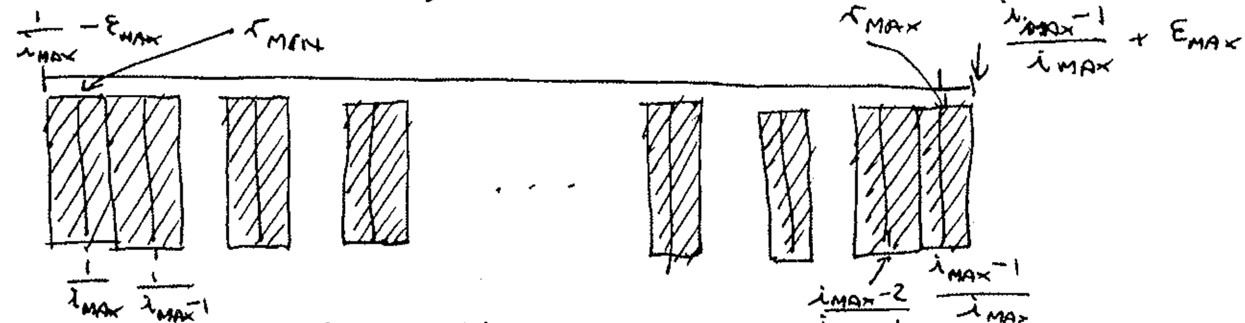
$$\frac{i_{MAX}-1}{i_{MAX}} - \frac{i_{MAX}-2}{i_{MAX}-1} = \frac{(i_{MAX}-1)^2 - (i_{MAX}-2)\,i_{MAX}}{i_{MAX}(i_{MAX}-1)}$$

$$= \frac{i_{MAX}^2 - 2\,i_{MAX} + 1 - i_{MAX}^2 + 2\,i_{MAX}}{i_{MAX}(i_{MAX}-1)} = \frac{1}{i_{MAX}(i_{MAX}-1)}$$

$$\frac{1}{i_{MAX}-1} - \frac{1}{i_{MAX}} = \frac{i_{MAX} - (i_{MAX}-1)}{i_{MAX}(i_{MAX}-1)} = \frac{1}{i_{MAX}(i_{MAX}-1)}$$ (4)

i.e. same expression)

(c)



Consider a zone of width $\pm \varepsilon_{MAX}$ around each rational.

Total length of number line $= \left(\frac{i_{MAX}-1}{i_{MAX}} + \varepsilon_{MAX}\right) - \left(\frac{1}{i_{MAX}} - \varepsilon_{MAX}\right)$

$$= \frac{i_{MAX} - 1 + \varepsilon_{MAX} i_{MAX}}{i_{MAX}} - \frac{1 - \varepsilon_{MAX} i_{MAX}}{i_{MAX}}$$

$$= \frac{i_{MAX} - 1 + \varepsilon_{MAX} i_{MAX} - 1 + \varepsilon_{MAX} i_{MAX}}{i_{MAX}}$$

$$= \frac{i_{MAX} - 2 + 2\varepsilon_{MAX} i_{MAX}}{i_{MAX}}$$

$$= \frac{i_{MAX} - 2}{i_{MAX}} + 2\varepsilon_{MAX}$$ (4)

Total "permissible" length of number line $= N_r \cdot 2\varepsilon_{MAX}$ (3)

i.e. for a uniform probability,

$$p = \frac{\text{total permissible length}}{\text{total length}} = \frac{2\varepsilon_{MAX} N_r}{\frac{i_{MAX}-2}{i_{MAX}} + 2\varepsilon_{MAX}}$$

(3)