

Densities and Central Properties of the Planets and the Moon (5)

Planet	Radius (equatorial km).	Density (g cm^{-3})	Central Pressure (Mbar)	Central Temperature ($^{\circ}\text{K}$)
Mercury	2440	5.43	~ 0.4	~ 2000
Venus	6042	5.20	~ 3	~ 5000
Earth	6378	5.515	3.6	6000
Moon	1738	3.34	0.045	~ 1800
Mars	3390	3.93	~ 0.4	~ 2000
Jupiter	71492	1.33	~ 80	~ 20000
Saturn	60268	0.69	~ 50	~ 10000
Uranus	25559	1.32	~ 20	~ 7000
Neptune	24766	1.64	~ 20	~ 7000

The Earth is differentiated, and the increase in density towards the centre just about compensates for our overestimate in gravity.

This method provides good estimates for relatively small bodies, with a uniform density, such as the Moon, which has a central pressure of only 45 kbar.

For Jupiter the central pressure is underestimated

Gravitational Potential Energy

(6)

The gravitational binding energy of a given distribution of matter is defined as the work done on the system to bring the matter "diffused" to infinity into the given distribution.

For a spherically symmetrical distribution of matter we suppose we have already brought from infinity an amount of matter $M(r)$.

The work done to bring an additional shell of matter $dM(r)$ is

$$\begin{aligned} \int_{\infty}^r \frac{GM(r)dM(r)}{r^2} dr &= -GM(r)dM(r) \int_r^{\infty} \frac{dr}{r^2} \\ &= -GM(r)dM(r) \left[-\frac{1}{r} \right]_r^{\infty} \\ &= \frac{-GM(r)dM(r)}{r} \end{aligned}$$

Note that we integrate only over r while $M(r)$ and $dM(r)$ are unchanged.

Hence the total gravitational binding energy E_B for the whole configuration after it has been built up to a radius R is

$$E_B = - \int_0^R \frac{GM(r)dM(r)}{r}$$

For uniform density ($\rho = \text{constant}$)

$$M(r) = \frac{4}{3}\pi r^3 \rho, \quad dM(r) = 4\pi r^2 \rho dr$$

$$E_B = - \int_0^R \frac{G \left(\frac{4}{3}\pi r^3 \rho \right) 4\pi r^2 \rho dr}{r}$$

$$= -G \left(\frac{4\pi \rho}{3} \right)^2 3 \int_0^R r^4 dr = -3G \left(\frac{4\pi \rho}{3} \right)^2 \frac{R^5}{5}$$

$$\therefore E_B = -\frac{3}{5} \frac{GM^2}{R}$$

For the Earth this is equal to -2.25×10^{32} Joules
This value differs by only 10% from the actual value
 -2.46×10^{32} Joules.

The density profile determines also the moment of inertia

$$I = \frac{8\pi}{3} \int_0^R r^4 \rho(r) dr = K MR^2$$

For a homogeneous body $K = \frac{2}{5} = 0.4$.
A core decreases its value, which for the Earth is 0.331 (21% less).