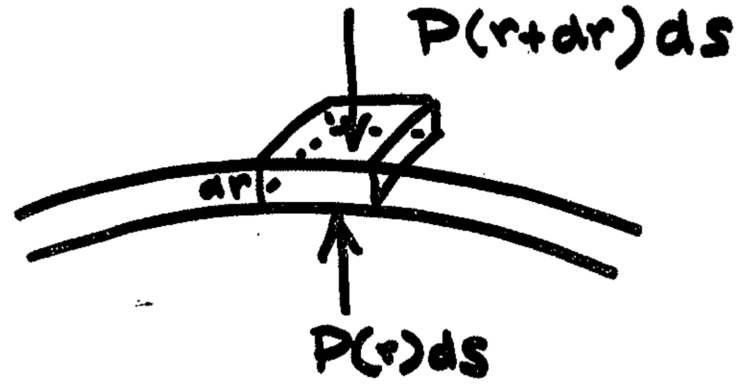
Hydrostatic equilibrium

Consider an infinitesimal volume element between r and $r+dr$ with base ds



If $P(r)$ is the total pressure on the bottom area and $P(r+dr)$ is the pressure on the top, the buoyant pressure force in the direction of increasing r is

$$P(r)ds - P(r+dr)ds$$

$$= P(r)ds - \left(P(r)ds + \frac{dP}{dr}dr\,ds\right)$$

$$= -\frac{dP}{dr}dr\,ds$$

This must be counteracted by the gravitational attraction to which the element of mass is subjected. The mass of the element is $\rho\,dr\,ds$. The attractive force is given by $\frac{GM(r)\rho\,dr\,ds}{r^2}$

For equilibrium

$$-\frac{dP}{dr}dr\,ds = \frac{GM(r)\rho\,dr\,ds}{r^2}$$

Therefore
$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho$$
This is the equation of hydrostatic equilibrium. Integrating from the centre to the surface

$$P(R) - P(0) = -\int_0^R \frac{G\,M(r)\,\rho(r)}{r^2}\,dr$$
But since at the surface $r = R$, $P(R) = 0$, the pressure at the centre is
$$P(0) = \int_0^R \frac{G\,M(r)\,\rho(r)}{r^2}\,dr .$$

For uniform density (ρ = constant), $M(r) = 4\pi/_3\, r^3 \rho$

$$P(0) = \frac{4\pi}{3} G\rho^2 \int_0^R r\,dr = \frac{4\pi}{3} G\rho^2 \left[\frac{r^2}{2}\right]_0^R$$
$$= \frac{2\pi}{3} G\rho^2 R^2 = \frac{3}{8\pi}\frac{GM^2}{R^4} ,$$

where $M = \frac{4\pi}{3} R^3 \rho$.

For the Earth $P(0) \simeq 1.7 \times 10^{11}$ Pa (1.7×10^{12} dynes/c
When a dense core is present as in all planet
the estimated value is $P_\oplus(0) \simeq 3.6 \times 10^{12}$ Pa

$\left\{ \begin{array}{l} (3.6 \times 10^{13}\ \text{dynes/cm}^2) \\ (3.6\ \text{MBar}) \end{array} \right.$