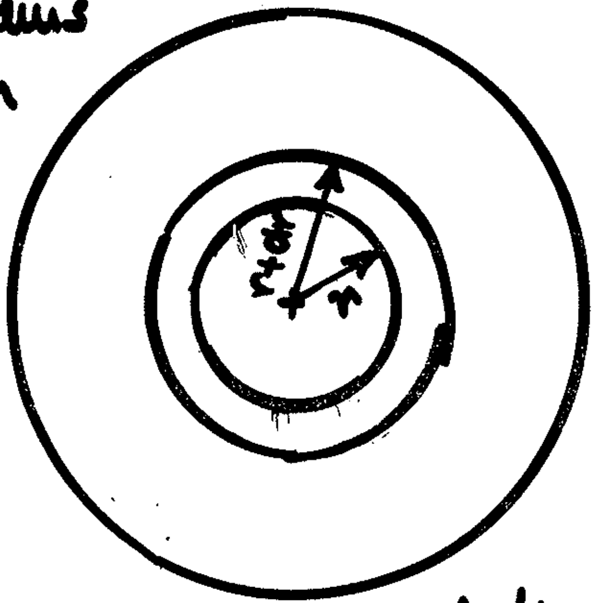
Conservation of mass

Let r be the radius measured from the centre.



Let $dM(r)$ be the mass of the shell of thickness dr.

If ρ is the density of the shell and $4\pi r^2 dr$ is its volume then

$$dM(r) = 4\pi r^2 \rho\, dr \quad \text{or} \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

Then $M(r)$ is the mass contained in the sphere of radius r then

$$M(r) = \int_0^r 4\pi r^2 \rho\, dr$$

If $r=R$ is the radius of the sphere then the total mass of the body is

$$M = M(R) = \int_0^R 4\pi r^2 \rho\, dr$$

The gravitational acceleration $g(r)$ at a distance r from the centre of the body has a radial inward direction and is equal to the one produced by a point mass $M(r)$ placed at the centre

$$g(r) = \frac{GM(r)}{r^2}$$

The acceleration

$$\underline{g} = -g\frac{\underline{r}}{r} = -g\hat{\underline{r}}$$

The gravitational potential energy per unit mass V is related to the acceleration by

$$\underline{g} = -\nabla V = -\frac{dV}{dr}\hat{r}$$

Therefore $\quad \dfrac{dV}{dr} = \dfrac{GM(r)}{r^2}$

If $r \geqslant R$, $M(r) = M$ = constant

therefore integrating

$$V = -\frac{GM}{r} + c$$

Taking $V = 0$ as $r \to \infty \Rightarrow c = 0$

$$V = -\frac{GM}{r}$$

At the boundary $r = R$ gives $V_1 = -\frac{GM}{R}$.