UNIVERSITY OF LONDON

BSc/MSci EXAMINATION April 2007

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MECHANICS & RELATIVITY

For First-Year Physics Students

Wednesday 25th April 2007: 10.00 to 12.00

Answer ALL parts of Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) A plane is flying horizontally at $250\,\mathrm{m\,s^{-1}}$ at an altitude of $300\,\mathrm{m}$ above the ground. The pilot must drop a load of supplies (initially at rest in the plane) so that it hits the ground at a specified target. Assuming air resistance can be neglected, calculate:

   (a) the horizontal distance from the target at which the load must be released, and,

   (b) the angle to the horizontal at which the load hits the ground.

   [6 marks]

(ii) An object of mass $2.0\,\mathrm{kg}$, initially at rest, is pushed with a constant horizontal force of $10.0\,\mathrm{N}$ across a horizontal surface. The coefficient of kinetic friction between the object and the surface is 0.4. Calculate:

   (a) the frictional force on the object,

   (b) the acceleration of the object

   (c) the total work done on the object in $1\,\mathrm{s}$.

   [6 marks]

(iii) A spring of spring constant k has a potential energy $U = \dfrac{1}{2}kx^2$, where x is the extension of the spring. Derive an expression for the force exerted by the spring.

   A spring of natural length $10\,\mathrm{cm}$ exerts a force of $0.8\,\mathrm{N}$ when extended to $12\,\mathrm{cm}$. Calculate the value of its spring constant k.

   [5 marks]

(iv) (a) A particle with momentum $\mathbf{p}$ has an instantaneous position vector $\mathbf{r}$. Write down an expression for $\mathbf{L}$, its angular momentum about the origin.

   (b) A force $\mathbf{F}$ is acting on the particle. Write down an expression for $\boldsymbol{\tau}$, the torque on the particle about the origin, and show that $\dfrac{d\mathbf{L}}{dt} = \boldsymbol{\tau}$.

   [5 marks]

(v) Write down an expression for the gravitational potential energy of an object of mass m on the surface of a planet of mass M and radius R. Hence find an expression for the escape velocity (i.e., the minimum speed needed to reach $r = \infty$) from the planet.

   Calculate the escape velocity from Mars (mass $= 6.42 \times 10^{23}\,\mathrm{kg}$ and radius $= 3.40 \times 10^6\,\mathrm{m}$).

   [5 marks]

[TOTAL 27 marks]

2. An unstable particle has a half-life $\tau_0 = 10^{-8}$ s in its rest frame. A beam of these particles is produced at a fixed point in the laboratory, and the particles travel at uniform speed u relative to the laboratory.

(i) If the number of particles in the beam has fallen to one half the original number after 10^{-6} s, calculate the speed u as a fraction of c.

[4 marks]

(ii) Calculate the fraction of the beam that remains at a distance of 3 km from the production point.

[2 marks]

(iii) The momentum of the particles in the beam is measured to be 14 GeV/c. Calculate (answer in MeV/c^2) the rest mass of the unstable particles.
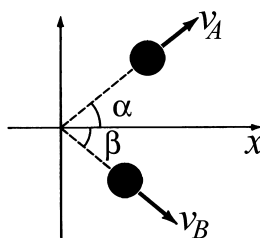
[3 marks]

[TOTAL 9 marks]

SECTION B

3. (i) Define the terms:

(a) *elastic collision*, and

(b) *inelastic collision*.

[4 marks]

(ii) A billiard ball initially moving in the $+x$ direction with speed u_A has an elastic collision with a second, initially stationary, billiard ball of the same mass. Immediately after the collision the first ball is moving at an angle α with a speed v_A, and the second ball is moving at an angle $-\beta$ with a speed v_B (see figure). Show that

$$v_A \cos\alpha + v_B \cos\beta = u_A$$
$$v_A \sin\alpha - v_B \sin\beta = 0$$
$$v_A^2 + v_B^2 = u_A^2$$

[6 marks]

(iii) Using the trigonometric identity

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \quad \text{[DO NOT PROVE]}$$

show that for the collision considered in part (ii):

$$v_A v_B \cos(\alpha + \beta) = 0 .$$

What can be concluded from this expression if:

(a) neither v_A nor v_B are zero, or

(b) the collision is one-dimensional (i.e., $\alpha = \beta = 0$)?

[8 marks]

(iv) In a given collision of the type discussed in parts (ii) and (iii) the first ball initially has a speed of $u_A = 2.0\,\mathrm{m\,s^{-1}}$, and is deflected in the collision by an angle $\alpha = 30°$. Calculate v_A, v_B, and β.

[8 marks]

(v) Suppose that instead of colliding elastically the balls stick together. Again assuming that before the collision the first ball has a speed of $u_A = 2.0\,\mathrm{m\,s^{-1}}$, while the second ball is stationary, determine the motion after the collision.

Given that the mass of a billiard ball is $0.17\,\mathrm{kg}$, calculate the decrease in total kinetic energy in this case. Where does the lost kinetic energy go?

[6 marks]

[TOTAL 32 marks]

4. (i) Show that the moment of inertia of a hollow cylinder (uniform density ρ, inner radius a, outer radius b, length l) about the axis of symmetry perpendicular to the flat ends (see diagram below) is given by

$$I = \frac{\pi \rho l}{2}(b^4 - a^4) .$$
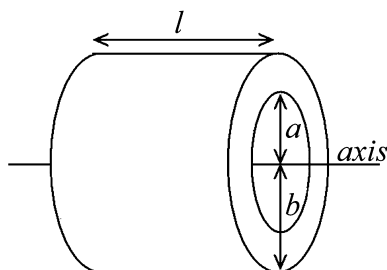
[7 marks]

(ii) Consider the following objects, each of mass M and radius R:

(a) a solid cylinder,

(b) a thin cylindrical shell,

(c) a flat disc.

Using the expression obtained in part (i), write down expressions for the moments of inertia of each of these objects in terms of M and R.

[6 marks]

(iii) A solid cylinder rolls from rest without slipping down a slope of vertical height H. Assuming friction and other dissipative effects can be neglected, use conservation of energy to show that when the cylinder reaches the bottom of the slope the speed of its centre of mass is given by

$$v = \sqrt{\frac{4gH}{3}}$$

where g is the acceleration due to gravity.

[7 marks]

(iv) A thin cylindrical shell of radius R rolls without slipping from rest down the same slope as considered in part (iii). Show that in this case the speed of its centre of mass at the bottom of the slope is given by

$$v = \sqrt{gH} .$$

[5 marks]

(v) A sealed empty can is made of thin sheet metal of areal density (i.e., mass per unit area) σ kg m^{-2}. It consists of a thin cylindrical shell of radius R and length l, with flat discs at the ends. There are no seams or other non-uniformities. Show that its moment of inertia is given by

$$I = \sigma \pi R^4 (2\alpha + 1)$$

where $\alpha = l/R$.

[7 marks]

[TOTAL 32 marks]

5. (i) A planet experiences a gravitational force from a star. Assuming that this is the only force acting on the planet, show that **L**, its angular momentum about the star, is constant.

[6 marks]

(ii) The planet follows an elliptical orbit around the star. Show that the rate at which a line joining the planet to the star sweeps out area is given by

$$\frac{dA}{dt} = \frac{L}{2m} = \text{constant}$$

where L is the magnitude of the angular momentum and m is the mass of the planet (i.e., Kepler's second law of planetary motion).

[8 marks]

(iii) Assuming now that the planet actually follows a *circular* orbit of radius R, show that the period of its orbit is given

$$T = \frac{2\pi R^{3/2}}{(GM)^{1/2}}$$

where M is the mass of the star, and G is the gravitational constant.

[7 marks]

(iv) Assuming that the Earth follows a circular orbit around the Sun (mass $1.99 \times 10^{30}\,\text{kg}$) of period 1 year, calculate the distance from the Earth to the Sun.

[3 marks]

(v) A binary star system consists of two stars, of masses M_1 and M_2, following circular orbits, of radii r_1 and r_2, about their common centre of mass. Write down Newton's second law for each of the stars separately, and, hence, show that the expression for the period, T, found in part (iii) can be used for the binary star system if R is replaced by the separation of the stars' centres and M is replaced by $M_1 + M_2$.

[8 marks]

[TOTAL 32 marks]

6. An observer at rest in frame S' moves with constant velocity $\mathbf{u}$ relative to an observer at rest in frame S, along the x-axis of the S-frame. A particle P moves with a velocity $\mathbf{v}$ in S and is observed in S' to have a velocity $\mathbf{v}'$.

(i) Write down appropriate Lorentz transformations for the space-time coordinates in frames S' and S and hence show that the components of velocity of P, measured by the observer at rest in S', are related to those measured in S through

$$
\begin{aligned}
v_x' &= \frac{v_x - u}{1 - uv_x/c^2} \\
v_y' &= \frac{v_y}{\gamma(1 - uv_x/c^2)}
\end{aligned}
$$

where $\gamma = (1 - u^2/c^2)^{-1/2}$.

[12 marks]

(ii) The speed of the frame S' relative to S is $u = 0.5c$. The observer at rest in S directs a laser beam at an angle $\theta_{\text{laser}} = 45°$ to the x-axis. Calculate the angle θ'_{laser} of the laser beam to the x'-axis as measured by the observer at rest in S'.

[10 marks]

(iii) Verify that the x' and y' components of the velocity of the photons in frame S' are consistent with the invariance of the speed of light.

[3 marks]

(iv) A straight rod, stationary in S, is oriented at an angle $\theta_{\text{rod}} = 45°$ to the x-axis. Show that the angle with respect to the x'-axis, measured in S' is given by

$$
\tan \theta'_{\text{rod}} = \gamma \tan \theta_{\text{rod}}
$$

and calculate the value of θ'_{rod}.

[4 marks]

(v) Explain why the values of θ'_{rod} and θ'_{laser} differ.

[3 marks]

[TOTAL 32 marks]

End