UNIVERSITY OF LONDON

BSc/MSci EXAMINATION April 2006

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MECHANICS & RELATIVITY

For First-Year Physics Students

Wednesday 26th April 2006: 10.00 to 12.00

Answer ALL parts of Question 1 and Question 2 in Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

SECTION A

1. (i) State Newton's first two laws of motion. When can the first law be regarded as a special case of the second?

[5 marks]

(ii) Two identical bodies move in one dimension with velocities v_1 and v_2. What is the total (Newtonian) kinetic energy, T? What is the velocity of the centre of mass? What is the kinetic energy of the centre of mass, T_{CM}?

[6 marks]

(iii) A snooker ball moving with velocity $v = 4\,\mathrm{m\,s^{-1}}$ collides obliquely and elastically with a stationary identical ball. The first ball scatters at an angle of 30^{o} at a speed of $2\sqrt{3}\,\mathrm{m\,s^{-1}}$. Calculate the speed and the angle at which the second ball recoils.

[5 marks]

(iv) Show that the *effective* potential $U^*(r)$ for a planet of mass m moving under the gravitational influence of the sun (mass M) is given by

$$U^*(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \ ,$$

where L is the planet's angular momentum. Show from this expression that the radius of a circular orbit is given by

$$r_{\mathrm{circ}} = \frac{L^2}{GMm^2} \ .$$

[6 marks]

(v) Show that for rotation about its symmetry axis, the moment of inertia of a uniform solid cylinder of radius a, height h and mass M, is $\frac{1}{2}Ma^2$. A skater is initially spinning on frictionless ice at 3 revolutions $\mathrm{s^{-1}}$ with her arms outstretched. By pulling in her arms, her effective radius (assuming her body and arms may be approximated by a uniform cylinder), is reduced by 25%. Calculate the new angular speed.

[5 marks]

[TOTAL 27 marks]

2. (i) State Galileo's principle of relativity. [2 marks]

(ii) State Einstein's postulates of special relativity. [3 marks]

(iii) State *three* consequences that follow from the Einstein postulates that would not follow from Galileo's principle and write a brief note of explanation on *one* of these. [4 marks]

[TOTAL 9 marks]

SECTION B

3. A cart of mass m slides without friction along a roller coaster track so that its potential energy over a certain region of the distance along the ground (x) is described by

$$U(x) = mg\left[h + \left(\frac{x^2}{3h^2} - 1\right)x\right],$$

where h is a constant distance and g is the acceleration due to gravity. The design is such that the cart cannot leave the track. Over the region of interest, $U(x) > 0$.

(i) Sketch $U(x)$ showing clearly the general shape of the curve and the values of x for the two equilibrium points. Indicate which equilibrium is stable and which is unstable. Derive expressions for the value of $U(x)$ at each equilibrium. [10 marks]

(ii) Taking $h = 10$ m, calculate the speed with which the cart must be moving as it passes through the stable equilibrium position if it is to escape the region under consideration. [7 marks]

(iii) Show that the period of small oscillations about the minimum is approximately given by $2\pi\sqrt{h/2g}$. Calculate this for the given quantities. [8 marks]

(iv) Suppose when the cart passes midway between the equilibria it has velocity $v = dx/dt$, and a total energy $U_0 + \delta$, where U_0 is the value of the potential energy at the *unstable* equilibrium position. Briefly describe qualitatively the subsequent motion for (a) $\delta > 0$, $v < 0$, (b) $\delta > 0$, $v > 0$, and (c) $\delta < 0$, $v > 0$. [7 marks]

[TOTAL 32 marks]

4. (i) A propeller pushes a boat with force F through water that resists the motion with a force $-\gamma v$ where v is the boat's velocity and γ is a positive constant. Write down Newton's second law for the boat's motion. Show that the steady velocity of the boat is given by $v_0 = F/\gamma$.

[4 marks]

(ii) If the boat's engine cuts out at time t_0 when it is at position s_0 travelling at velocity v_0, show that the subsequent motion of the boat is described by

$$v = v_0 \exp\left[-\frac{\gamma}{m}(t - t_0)\right] , \quad (4.1)$$

$$s = s_0 + \frac{m v_0}{\gamma}\left\{1 - \exp\left[-\frac{\gamma}{m}(t - t_0)\right]\right\} \quad (4.2)$$

[12 marks]

(iii) Three bandits each of mass 75 kg steal gold bars of total mass 50 kg. Fleeing chasing police, they come to a boat repair yard, get into a boat of mass 450 kg, start the motor, and head off, the boat quickly reaching a steady speed of $20\,\mathrm{m\,s^{-1}}$ in the still water. Being under repair, however, the boat is faulty. The engine cuts out after 300 s, the motion then being described by equations (4.1) and (4.2). Draw a graph showing $v(t)$ and $s(t)$ indicating clearly the period for which the motor operates and calculate the values of v and s as $t \to \infty$. (Take $\gamma = 3\,\mathrm{kg\,s^{-1}}$.)

[8 marks]

(iv) A few minutes after the bandits start their boat, three policemen (each of mass 75 kg) pursue the bandits in a faulty boat of the same mass, that cuts out after running for 450 s. The boat's propeller is 20% less powerful than the bandits' boat (i.e. it is driven by a force, F, that is 20% smaller). Will they catch up with the bandits? Would the outcome be different if instead four policemen give chase? Explain your answers.

[8 marks]

[TOTAL 32 marks]

5. A rocket containing nuclear waste of mass m_0 is launched tangentially to the Earth's orbit, so that after the launch it travels at the Earth's orbital speed, v_E. It is proposed to dispose of the waste by reducing the rocket to rest in the solar system frame so that it then falls into the Sun. Ignore the effects of the Earth's gravity.

(i) Calculate the Earth's orbital speed v_E assuming a circular orbit about the Sun of radius $r_E = 1.5 \times 10^{11}$ m. [4 marks]

(ii) Using the rocket equation $v_{final} - v_{initial} = v_{ex} \ln \left(m_{initial}/m_{final}\right)$, where $| v_{ex} |$ is the speed with which the fuel is ejected relative to the rocket, calculate the minimum fuel mass required to bring the waste to rest, expressing your answer as a fraction of m_0. (Take $v_{ex} = -3\,\mathrm{km\,s^{-1}}$, and ignore the mass of the rocket structure.) [8 marks]

(iii) The rocket velocity is instead *increased* from v_E to v_1 (i.e. now $v_{ex} = +3\,\mathrm{km\,s^{-1}}$). The orbit is now an ellipse in which the closest approach to the Sun (perihelion) is r_E, and the furthest distance from the Sun (aphelion) is r_2. Show that v_2, the speed at aphelion, is related to v_1, by $v_2 = \beta v_1$, where $\beta = (r_E/r_2)$. [2 marks]

(iv) By using the conservation of mechanical energy show that

$$v_1^2 = \frac{2GM_{sun}}{r_E(1+\beta)} ,$$

where M_{sun} is the mass of the Sun. Given that $v_E^2 = GM_{sun}/r_E$ show that

$$v_1^2 = \frac{2v_E^2}{1+\beta} .$$

[8 marks]

(v) Calculate the value of v_1 and v_2 if $\beta = 0.2$. [2 marks]

(vi) The rocket is brought to rest at aphelion. Given that the rocket started near the Earth with velocity v_E, what is the *total* boost required to bring it to rest again? [2 marks]

(vii) Calculate the total fuel now required to bring the rocket to rest as a fraction of m_0. [6 marks]

[TOTAL 32 marks]

6. The momentum and energy of a particle of mass m, moving at speed $\mathbf{v}$, are defined via

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \ .$$

(i) Show that the particle velocity can be written $\mathbf{v} = \mathbf{p}c^2/E$.
Hence prove the relativistic energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4 \ .$$

What is the energy-momentum relation for a photon?

[12 marks]

(ii) The energy and momentum of a particle, measured by an observer at rest in an inertial frame S', are related to those measured by an observer at rest in S by the Lorentz transformations:

$$p'_x = \gamma\left(p_x - \beta\frac{E}{c}\right), \quad p'_y = p_y, \quad p'_z = p_z \ , \quad \frac{E'}{c} = \gamma\left(\frac{E}{c} - \beta p_x\right),$$

where S' moves relative to S in the positive x-direction at a velocity $u = \beta c$, and $\gamma = (1 - \beta^2)^{-1/2}$.
The observer in S' carries a laser that emits photons of energy E' and moves towards the observer in S. Show that the energy, E, as measured by the observer in S, is given by

$$E = E'\sqrt{\frac{1+\beta}{1-\beta}} .$$

[12 marks]

(iii) Neutral hydrogen atoms emit a radio-frequency spectral line at 1420 MHz. A cloud of hydrogen gas in outer space is observed to emit this line but its frequency is shifted by −500 kHz. Estimate the magnitude and direction of the cloud's velocity along the line-of-sight with respect to the Earth.

[8 marks]

[TOTAL 32 marks]

End