UNIVERSITY OF LONDON

BSc/MSci EXAMINATION April 2005

for Internal Students of Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant Examination for the Associateship

MECHANICS & RELATIVITY

For First-Year Physics Students

Wednesday 27th April 2005: 10.00 to 12.00

Answer ALL parts of Question 1 and Question 2 in Section A and TWO questions from Section B.

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Write your CANDIDATE NUMBER clearly on each of the FOUR answer books provided.

If an electronic calculator is used, write its serial number in the box at the top right hand corner of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the horizontal box on the front cover of its corresponding answer book.

Hand in FOUR answer books even if they have not all been used.

You are reminded that the Examiners attach great importance to legibility, accuracy and clarity of expression.

Turn over for questions

SECTION A

1. (i) State Newton's *Third Law of Motion*. Two astronauts are taking a "space-walk" in a region of space where all external forces can be ignored. Show that, if they interact only by push-pull forces that form an "action-reaction" pair then the total linear momentum of the two astronauts is constant.

[5 marks]

(ii) A cricket ball is thrown upwards at an angle α with a horizontal playing field at an initial speed u. Find expressions for the time t_h that the ball takes to reach its maximum height and the time t_r that the ball takes to return to the field assuming that g is constant and the effect of air resistance can be ignored. Show that $t_r = 2t_h$, whatever the angle α.

[4 marks]

(iii) The gravitational force of the Earth, mass M_E, on a body of mass m at a distance r from the centre of the Earth, can be represented by the potential energy function:

$$U(r) = -\frac{GmM_E}{r}$$

where G is the gravitational constant. Use the conservation of mechanical energy to find an expression for the *escape speed* υ_e, the speed that the body must have on the Earth's surface at radius R_E to escape from the Earth's gravitational field. Re-express your solution for υ_e in terms of g, the acceleration due to gravity at the Earth's surface and R_E.

[5 marks]

(iv) A rocket travelling at an initial velocity of $4\,\mathrm{km\,s^{-1}}$ in free space accelerates to a final velocity of $8\,\mathrm{km\,s^{-1}}$ by ejecting fuel at a constant speed of υ_{ex} relative to the rocket. If the fuel constituted 80% of the initial mass of the rocket and all the fuel was burnt to achieve this acceleration, calculate the constant speed υ_{ex}.

[4 marks]

(v) State the relationship between the *torque* of the force on a rigid body and the *angular momentum* of the body about a fixed point. Define a *central force*. Show that if the only force acting on a body is a central force then angular momentum is conserved.

[4 marks]

(vi) A flywheel in the shape of uniform circular disc has moment of inertia $16\,\mathrm{kg\,m^2}$ about an axis through the centre of the disc and perpendicular to the plane of the disc. The flywheel is rotating at 5 cycles per second when 4000 J of energy is extracted to provide emergency power. What is the new rotation speed of the flywheel in cycles per second after the power has been extracted? What is the constant power that must be supplied to the flywheel to take it back to its initial rotation speed in 5 seconds?

[5 marks]

[TOTAL 27 marks]

2. Consider two inertial frames S and S'. At the instant $t = t' = 0$, the coordinate axes of S and S' coincide. S' moves with constant velocity relative to S; the relative velocity, $\boldsymbol{u}$, is parallel to the x axis in S. The Lorentz transformations of space and time may be written:

$$ct' = \gamma\,[ct - \beta x] \quad x' = \gamma\,[x - \beta ct] \quad y' = y \quad z' = z \ ;$$

where $\beta = u/c$, $\gamma = 1\Big/\sqrt{1-\beta^2}$, c is the speed of light and the other symbols have their usual significance.

A body is observed in S to move such that its position at time t is $x = vt$, $y = \frac{1}{2}at^2$ and $z = 0$.

(i) Show that, when observed in S', the x' position of the body at time t' is

$$x' = \left[\frac{v-u}{1-\beta\frac{v}{c}}\right] t'\,.$$

[6 marks]

(ii) Further, show that the y' position of the body at time t' is

$$y' = \tfrac{1}{2}a\left[\frac{1}{\gamma\left(1-\beta\frac{v}{c}\right)}\right]^2 t'^2\,.$$

[3 marks]

[TOTAL 9 marks]

SECTION B

3. The interaction between two *inert-gas* atoms, which attract with the *van der Waals* force, is often represented by a potential function known as the *Lennard-Jones* potential, which is given by:

$$U(r) = \varepsilon\left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^{6}\right]$$

where a and ε are positive constants and the atoms are distance r apart.

(i) Show that the *equilibrium separation* between the two atoms occurs when $r = r_0 = a$.
[4 marks]

(ii) What is the value of the potential function at the equilibrium position r_0?
[2 marks]

(iii) Sketch the potential function $U(r)$ and its two component terms, for values of $r > 0$. Indicate the points that correspond to the position of the equilibrium separation and the value of the potential function at equilibrium. Which part of the potential function corresponds to a repulsive force and which part corresponds to an attractive force?
[7 marks]

(iv) Argon is an inert-gas atom, which interacts with other Argon atoms through a potential of the Lennard-Jones form. Two such atoms can form a molecule if the temperature is low enough. Two Argon atoms in a molecule have kinetic energy due to their internal motion $K_0 = \varepsilon/4$ at the equilibrium position. Express the total mechanical energy in terms of ε and sketch the level on your figure. How much work is done by an external force that *dissociates* the molecule? (i.e. a force that separates the two Argon atoms an infinite distance apart).
[5 marks]

(v) Determine the second derivative of the potential function and hence find the angular frequency ω of small amplitude oscillations about the equilibrium position

$$\omega = \sqrt{\frac{\left(\frac{d^2U(r)}{dr^2}\right)_{r_0}}{\mu}},$$

in terms of a, ε and the *reduced mass* of the two-atom system μ.
[6 marks]

(vi) What is the *reduced mass* μ of a molecule formed from two Argon atoms each of mass $40m_H$ where m_H is the mass of a hydrogen atom?
[3 marks]

(vii) Calculate the *period* of small amplitude oscillations of an Ar_2 molecule bound by a Lennard-Jones potential with $a = 3.4 \times 10^{-10}$ m, $\varepsilon = 1.7 \times 10^{-21}$ J, given that $m_H = 1.67 \times 10^{-27}$ kg.
[5 marks]

[TOTAL 32 marks]

4. (i) What condition must hold in a two-body scattering process if *conservation of linear momentum* is to apply? What condition must hold in a two-body elastic scattering process if we are able to ignore potential energy in applying the *conservation of mechanical energy*?

[4 marks]

(ii) During a game of snooker a student has the opportunity to win with only the pink and black balls left on the table. The white (or cue) ball can knock the pink ball into a pocket if the pink ball, which is at rest on the table, is struck head-on so that the pink ball continues in the same straight line as the white ball had initially. Assume that the white and the pink ball have equal mass, the collision is elastic and the white ball is struck with a constant speed u in the x-direction (the line between the white and pink balls). Use kinetic energy and momentum conservation to show that the speed v_p with which the pink ball recoils in the x-direction is equal to u, the speed the white ball had initially, while the white ball is reduced to rest.

[12 marks]

(iii) The student now finds that the white ball is well placed to knock the black ball (also at rest on the table and of the same mass as the white ball) into another pocket if the black ball can recoil at an angle between 67° and 68° with the line between the white and black balls. The student propels the white ball with a constant speed of $2\,\mathrm{ms}^{-1}$ to give the black ball a glancing blow. The white ball is observed to scatter with a speed of $1.85\,\mathrm{ms}^{-1}$ at an angle of 22.3° with the original line between white and black balls, on the other side of the table to the pocket. Use kinetic energy conservation to find the speed of recoil of the black ball. Then use conservation of momentum perpendicular to the line between the white and black balls to see if the student is successful.

[13 marks]

(iv) The angles that the recoiling black ball and the scattered white ball make with the initial direction of white ball should satisfy a simple relationship approximately. What is that relationship? Name two conditions that must apply to the scattering process for this simple relationship to hold.

[3 marks]

[TOTAL 32 marks]

5. A space-probe of mass m is to make a voyage from the Earth (at distance r_E from the Sun) to Venus (at distance r_V from the Sun).

(i) By expressing conservation of mechanical energy for the probe in terms of its radial and transverse components of velocity υ_r and υ_t respectively and using angular momentum conservation, show that the total energy of the probe, moving in the gravitational force (gravitational constant G) of the Sun (mass M_S) with constant angular momentum L at radial distance r from the sun is given by:

$$E = \frac{1}{2}m\upsilon_r^2 + \frac{L^2}{2mr^2} - \frac{GmM_S}{r}\,.$$

You may ignore the gravitational effects of Earth and Venus on the probe. [5 marks]

(ii) The orbit usually chosen for a space-probe is an ellipse such that at *aphelion*, where r has its maximum value r_{max}, r is equal to the radius r_E of the Earth's orbit round the sun (assumed circular) and at *perihelion*, where r has its minimum value r_{min}, r is equal to the radius r_V of Venus orbit round the Sun (also assumed to be circular). Sketch the shape of the space-probe orbit, paying particular attention to the shape of the orbit close to the circular orbits of the Earth and Venus, which should also be sketched. Indicate the directions of the transverse velocity of the probe υ_{tE} at aphelion and the transverse velocity of the probe υ_{tV} at perihelion. [6 marks]

(iii) Use angular momentum conservation to find a relationship between the initial transverse velocity of the probe relative to the Sun υ_{tE} at aphelion at the Earth orbit, the final transverse velocity of the probe relative to the Sun υ_{tV} at perihelion at Venus and the radii of the orbits of the Earth r_E, and Venus r_V. [3 marks]

(iv) Use energy conservation for the elliptical orbit and the direction of the probe velocity at aphelion and perihelion to show that

$$\frac{1}{2}m\upsilon_{tE}^2 - \frac{GmM_S}{r_E} = \frac{1}{2}m\upsilon_{tV}^2 - \frac{GmM_S}{r_V}\,.$$

[4 marks]

(v) Use the result of angular momentum conservation in (iii) and the expression in part (iv), to show that υ_{tE}, the initial tangential velocity of the probe relative to the sun, is given by

$$\upsilon_{tE}^2 = \left(\frac{r_V}{r_E}\right)\left(\frac{2GM_S}{r_E + r_V}\right).$$

[Hint: express the difference between the squares of the radii of the orbits of Venus and the Earth as the product of their sum and the difference.]

[7 marks]

(vi) Use the planets' orbital radii $r_E = 1.50 \times 10^{11}$ m and $r_V = 1.08 \times 10^{11}$ m and values $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ and $M_S = 1.99 \times 10^{30}$ kg to find the initial tangential velocity of the probe υ_{tE} relative to the Sun.

[3 marks]

(vii) If the probe starts from an orbit around the Earth it will already have a tangential velocity around the sun approximately equal to the orbital velocity of the Earth round the sun. Estimate this tangential velocity given that the Earth takes 1 year ($\approx \pi \times 10^7$ s) to orbit the Sun. Hence calculate the change in speed the probe must undergo to enter the elliptical orbit that will take it to Venus. In which direction must the probe rocket motor be fired?

[4 marks]

[TOTAL 32 marks]

6. Consider two inertial frames S and S'. At the instant $t = t' = 0$, the coordinate axes of S and S' coincide. S' moves with constant velocity relative to S; the relative velocity, $\boldsymbol{u}$, is parallel to the x-axis in S. The Lorentz transformations of time and space may be written:

$$ct' = \gamma\,[ct - \beta x] \quad x' = \gamma\,[x - \beta ct] \quad y' = y \quad z' = z \ ,$$

where $\beta = u/c$, $\gamma = 1\Big/\sqrt{1-\beta^2}$, c is the speed of light and the other symbols have their usual significance. The corresponding transformations of energy and momentum may be written:

$$E' = \gamma\,[E - \beta c p_x] \quad cp'_x = \gamma\,[cp_x - \beta E] \quad cp'_y = cp_y \quad cp'_z = cp_z \ .$$

(i) A distant star recedes rapidly from an observer assumed to be stationary on Earth. The pattern of spectral lines arriving at Earth from the star is observed to be shifted to longer wavelengths, an effect known as the 'red shift'. Give a brief explanation of the origin of the red shift.

[6 marks]

(ii) Show that the red shift, R_S , for a star receding at speed u is

$$R_S = \frac{f'}{f} = \sqrt{\frac{1-\beta}{1+\beta}}$$

where f is the frequency at which a particular spectral line is measured by an observer stationary with respect to the light source and f' is the frequency of the same spectral line measured by an observer moving at a speed u with respect to the source. You may assume that the energy of a photon is related to its frequency by Plank's constant, h, i.e. that $E_\gamma = hf$, where E_γ is the energy of the photon.

[14 marks]

(iii) The Virgo nebula is 7.5 million light years from Earth. The observed red shift of the light from Virgo is $R_S = 0.999$. At what speed is Virgo receding from Earth?

[7 marks]

(iv) The red shift for the Hydra nebula is 0.975. How far is Hydra from Earth? You may assume that the Hubble law $u = Hd$ (where u is the velocity at which a distant source is receding, H is the Hubble constant and d is the distance to the source) holds.

[5 marks]

[TOTAL 32 marks]

End