UNIVERSITY OF LONDON [MP1 2006]

B.Sc. and M.Sci. DEGREE EXAMINATIONS 2006

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M. PHYS 1

Date: Thursday 27th April 2006 Time: 10 am - 1 pm

Do not attempt more than SIX questions

Please use a separate answerbook for each question.

[*Before starting, please make sure that the paper is complete; there should be 6 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

[MP1 2006]

1. (i) Evaluate the limits:

(a)
$$\lim_{x \to 0} \frac{\sinh\,(e^x - 1)}{x} \;;$$

(b)
$$\lim_{x \to \infty} \frac{(x+1)^{1/2} \;-\; (x-1)^{1/2}}{(x+2)^{1/2} \;-\; (x-2)^{1/2}} \;.$$

(ii) Sketch the curve described by

$$y^2 \;=\; \frac{(x+1)\,(x-3)}{(x+4)} \;,$$

determining the zeros, asymptotes, large x behaviour and regions where the curve does not exist.

2. (i) Evaluate the integral

$$\int_0^1 x^3\, e^{-x^2}\; dx \;.$$

(ii) Using the substitution $t = \tan\; x/2$, or otherwise, show that

$$\int_0^{\pi/2} \frac{dx}{2 + \sin\, x} \;=\; \frac{\pi}{3\sqrt{3}} \;.$$

(iii) Using the substitution $x = \sinh z$, or otherwise, show that the integral

$$I \;=\; \int_0^1 (1 + x^2)^{1/2}\; dx \;,$$

is equal to

$$I \;=\; 2^{-1/2} \;+\; \frac{1}{2}\; \sinh^{-1} 1 \;.$$

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3. (i) Consider the function, written in polar co-ordinates

$$r = \cos\theta , \tag{1}$$

with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(a) Determine the path length of the curve.

(b) Convert the polar equation (1) above into Cartesian form (i.e. in terms of x and y) and hence show it describes a circle of radius $1/2$.

(ii) Find the centre of mass of an equilateral triangle with vertices at $(-1/2,\ 0)$, $(1/2,\ 0)$ and $(0,\ \sqrt{3}/2)$.

(iii) Find the first and second partial derivatives of

$$u(x,\ y) = \sin\left(\frac{y}{x}\right) .$$

4. Prove that $(\cos\theta + i\,\sin\theta)^n = \cos n\theta + i\sin n\theta$ (De Moivre's Theorem).

If $z_1 = 3 - 4\,i$ and $z_2 = -\sqrt{3} + i$,

(i) find the real and imaginary parts of z_1^{-1} ;

(ii) find the moduli and arguments of z_1, z_2 and $z_2 \,/\, z_1$;

(iii) find the modulus and argument of z_2^7 ;

(iv) find the moduli and arguments of all values of $z_1^{\frac{1}{2}}$;

(v) plot the results of (iv) on a rough sketch of the complex plane.

Quote arguments in radians; values in the range $-\pi < \theta < 2\pi$ are acceptable.

5. (i) Find unit normal vectors $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ to the two planes $x+2y-z = -1$ and $2x-y+3z = 3$.

(ii) Find a unit vector directed along the line of intersection of the two planes.

(iii) Find the coordinates of the point where the line of intersection cuts the $z=0$ plane. Hence obtain the vector equation of the line of intersection.

(iv) Given a plane with equation

$$2x + y + z = \alpha ,$$

where α is a constant, find the normal vector to the plane and show that it is co-planar with $\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{n}}_2$ of part (i).

(v) Given that the line of intersection obtained in part (ii) lies in this plane, find the constant α.

6. The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are related by $\boldsymbol{u} = \boldsymbol{T}\boldsymbol{v}$ and $\boldsymbol{w} = \boldsymbol{T}(\boldsymbol{T}\boldsymbol{v}) \equiv \boldsymbol{T}^2\boldsymbol{v}$, where the 2×2 matrix

$$\boldsymbol{T} = \begin{pmatrix} 5 & 8 \\ 2 & -1 \end{pmatrix} .$$

(i) Find $\boldsymbol{u}$ and $\boldsymbol{w}$ when $\boldsymbol{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(ii) Find a matrix S such that $ST = I$.

(iii) Write down the definitions of the eigenvalues and eigenvectors of T and find them.

(iv) Write down the eigenvalues and eigenvectors of T^2 and S and state how they are related to those of T.

7. Consider the differential equation

$$\frac{dx}{dt} + 2x = A .$$

(i) Prove that, when A is a constant, the solution to the equation is

$$x(t) = x(0)\, e^{-2t} + \frac{1}{2} A\,(1 - e^{-2t}) .$$

(ii) For $A = 4$ and $x(0) = 1$, draw a rough graph of x as a function of t, indicating the asymptote as $t \to \infty$.

(iii) Now consider the scenario where $A = 4$ for $t \leq 1$, but $A = 0$ for $t > 1$. For arbitrary $x(0)$, obtain an expression for $x(t)$ when $t > 1$.

(iv) Use the result of part (iii) to show that $x(2) = x(0)$ when

$$x(0) = 2 \left(\frac{e^2 - 1}{e^4 - 1} \right) .$$

(v) Evaluate the expression for $x(0)$ in part (iv) and draw a rough graph of $x(t)$ in the range $0 < t < 2$ under the scenario of part (iii).

8. The equation of a plane is given by

$$a\,x + b\,y + c\,z = d ,$$

where

$$\boldsymbol{n} = a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}$$

is the normal vector of the plane.

(i) Determine the equation of the tangent plane of the $x^2 + y^2 + z^2 = 9$ surface at the point $(2, -1, 2)$.

(ii) Determine the cosine of the angle this tangent plane makes with the surface $z = x + y^2 - 1$ at the point $(2, -1, 2)$.

9. (i) State Stokes's theorem. Describe the regions over which the integrations are carried out and the quantities that are being integrated.

(ii) Evaluate the line integral

$$\oint_{\mathcal{C}} \boldsymbol{V} \cdot d\boldsymbol{r} \ ,$$

where $\mathcal{C}$ is a closed path consisting of the straight lines from the point $(1, 0)$ to $(0, 1)$, the point $(0, 1)$ to $(-1, 0)$ and finally a semi-circle connecting the points $(-1, 0)$ and $(1, 0)$ running via the third and fourth quadrants and

$$\boldsymbol{V} \ = \ (x^2 z \ + \ y^2)\,\boldsymbol{i} \ - \ xy\,\boldsymbol{j} \ + \ z^2\boldsymbol{k} \ .$$

(iii) Check this result by using Stokes's theorem written for the interior of $\mathcal{C}$ and taking $\boldsymbol{k}$ as the normal direction to the surface $\mathcal{S}$ capping the path $\mathcal{C}$.

10. (i) State the divergence theorem. Describe the regions over which the integrations are carried out and the quantities that are being integrated.

(ii) Consider the vector field

$$\boldsymbol{V} \ = \ \frac{x\,\boldsymbol{i} \ + \ y\,\boldsymbol{j} \ + \ z\,\boldsymbol{k}}{\sqrt{x^2 \ + \ y^2 \ + \ z^2}} \ .$$

Find the outward flux across the boundary of the hemisphere bounded by the spherical surface $x^2 + y^2 + z^2 = R^2$ (for $z > 0$) and the xy-plane.

(iii) Use the divergence theorem to verify the result.

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