$\underset{\sim}{F} = -\nabla V$ (force = − gradient of potential) ①

In spherical polars: $\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \psi}, \frac{1}{r \sin\psi}\frac{\partial}{\partial \phi} \right)$

$$\dot{E} = \frac{d}{dt}\left(\tfrac{1}{2} I \Omega^2 - \frac{G m_p m_s}{2a} \right) = I \Omega \dot{\Omega} + \frac{G m_p m_s}{2a^2} \dot{a}$$

But $G(m_p + m_s) = n^2 a^3$ or $G = \dfrac{n^2 a^3}{m_p + m_s}$

i.e. $$\frac{G m_p m_s \dot{a}}{2a^2} = \frac{n^2 a^3}{m_p + m_s} \cdot \frac{\dot{a}}{2a^2} \cdot \frac{m_p m_s}{1} = \frac{1}{2} \frac{m_p m_s}{m_p + m_s} n^2 a \dot{a}$$

i.e. $$\dot{E} = I \Omega \dot{\Omega} + \frac{1}{2} \frac{m_p m_s}{m_p + m_s} n^2 a \dot{a}$$ (< 0 because energy is dissipated as heat)

Angular momentum:

$$L = I\Omega + \frac{m_p m_s}{m_p + m_s} a^2 n \quad ; \quad \dot{L} = 0$$ (ang. mom. is conserved)

(ROT. ANG. MOM. + ORB. ANG. MOM)

i.e. $$I \dot{\Omega} = - \frac{m_p m_s}{(m_p + m_s)} \frac{d}{dt}(a^2 n)$$

$$\frac{d}{dt}(a^2 n) = 2a \frac{da}{dt} n + a^2 \frac{dn}{dt}$$; but $n^2 a^3 = \text{const.}$ (Kepler's 3rd law)

i.e. $2n \frac{dn}{dt} a^3 + 3n^2 a^2 \frac{da}{dt} = 0$ i.e. $\dfrac{dn}{dt} = -\dfrac{3}{2} \dfrac{n}{a} \dfrac{da}{dt}$

Hence $$\frac{d}{dt}(a^2 n) = \left(2an + a^2 \left(-\frac{3}{2}\frac{n}{a} \right) \right) \frac{da}{dt} = \left(2an - \frac{3}{2} an \right) \dot{a}$$

$$= \frac{1}{2} a n \dot{a}$$

i.e. $$I \dot{\Omega} = -\frac{1}{2} \frac{m_p m_s}{m_p + m_s} n a \dot{a}$$

Hence $$\dot{E} = \Omega\left(-\frac{1}{2}\frac{m_p m_s}{(m_p+m_s)} n a \dot{a}\right) + n\left(\frac{1}{2}\frac{m_p m_s}{(m_p+m_s)} n a \dot{a}\right)$$

$$= -\frac{1}{2}\frac{m_p m_s}{m_p+m_s} n a \dot{a}\,(\Omega - n) \qquad (< 0)$$

Hence $\text{sign}(\dot{a}) = -\text{sign}(\dot{\Omega}) = \text{sign}(\Omega - n)$

(from express. for $I\dot{\Omega}$)

Note that $\dot{E} = -\frac{1}{2}\frac{m_p m_s}{m_p+m_s} n a \dot{a}(\Omega - n)$ and (for satellites outside synchronous orbit) $\dot{E} = -\Gamma(\Omega - n)$

Hence $$\Gamma = \frac{1}{2}\frac{m_p m_s}{m_p+m_s} n a \dot{a}$$

$$\Gamma = +m_s \frac{\partial V_{NC,EXT}}{\partial \psi} \quad ; \quad \zeta = \frac{m_s}{m_p}\left(\frac{R_p}{a}\right)^3 R_p \quad ; \quad g = \frac{G m_p}{R_p^2} \, ;$$

$$\frac{\partial}{\partial \psi}\left(P_2(\cos\psi)\right) = -\frac{3}{2}\sin 2\psi \quad ; \quad V_{NC,EXT} = -k_2 \zeta g \left(\frac{C}{r}\right)^3 P_2(\cos\psi)$$

Hence $$\Gamma = \frac{3}{2} m_s k_2 \zeta g \left(\frac{C}{r}\right)^3 \sin 2\psi$$

$$= \frac{3}{2} m_s k_2 \underbrace{\left[\frac{m_s}{m_p}\left(\frac{C}{a}\right)^3 C\right]\left[\frac{G m_p}{C^2}\right]}_{\text{using } R_p = C} \underbrace{\left(\frac{C}{a}\right)^3}_{\substack{\text{using} \\ r = a}} \sin 2\psi$$

$$= \frac{3}{2} k_2 \frac{G m_s^2}{a^6} C^5 \sin 2\varepsilon$$ (using $\psi = \varepsilon$ for lag angle; N.B. $2\varepsilon \approx \frac{1}{Q}$)

Hence $$\Gamma = \frac{1}{2}\frac{m_p m_s}{(m_p+m_s)} n a \dot{a} = \frac{3}{2} k_2 \frac{G m_s^2}{a^6} C^5 \sin 2\varepsilon$$

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$$\text{i.e. } \dot{a} = \text{sign}(\Omega - n)\, 3k_2 \frac{G m_s^2}{a^6} c^5 \sin 2\varepsilon \cdot \frac{(m_p + m_s)}{m_p m_s n a}$$

$$= \text{sign}(\Omega - n) \frac{3k_2}{Q} \cdot G(m_p + m_s) \frac{m_s}{m_p} \left(\frac{c}{a}\right)^5 \cdot \frac{1}{n a^2}$$

$$= \text{sign}(\Omega - n) \frac{3k_2}{Q} \cdot n^2 a^3 \frac{m_s}{m_p} \left(\frac{c}{a}\right)^5 \cdot \frac{1}{n a^2}$$

$$= \text{sign}(\Omega - n) \frac{3k_2}{Q} \frac{m_s}{m_p} \left(\frac{c}{a}\right)^5 n a \qquad \left(\text{using } G(m_p + m_s) = n^2 a^3\right)$$

$$n = \left(\frac{G(m_p + m_s)}{a^3}\right)^{1/2} \approx \left(\frac{G m_p}{a^3}\right)^{1/2}$$

$$\text{i.e. } \dot{a} \approx \frac{3k_2}{Q} \frac{m_s}{m_p} \left(\frac{G m_p}{a^3}\right)^{1/2} \left(\frac{c}{a}\right)^5 a = \frac{3k_2}{Q} m_s \left(\frac{G}{m_p}\right)^{1/2} c^5 a^{-11/2}$$

with solution

$$\frac{2}{13} a_0^{13/2} \left[1 - \left(\frac{a_i}{a_0}\right)^{13/2}\right] = \frac{3k_2}{Q} \left(\frac{G}{m_p}\right)^{1/2} c^5 m_s \Delta t$$
