B.Sc. EXAMINATION

MAS322 Relativity

7 May 2008, 14:30
Time Allowed: 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

You are reminded of the following information, which you may use without proof.

The metric tensor of special relativity is η_{ab} such that

$$ds^2 = \eta_{ab} dx^a dx^b = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

The Lorentz transformations between two frames F and F' in standard configuration are

$$x' = \gamma(x - Vt), \quad t' = \gamma\left(t - \frac{Vx}{c^2}\right), \quad y' = y, \quad z' = z$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$ and F' is moving with speed V relative to F.

Partial derivatives:

$$Q_{,a} = \frac{\partial Q}{\partial x^a}$$

Covariant derivatives are denoted by semicolon: e.g. $V_{a;b} = V_{a,b} - \Gamma^c{}_{ab} V_c$

The connection:

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left[g_{db,c} + g_{dc,b} - g_{bc,d} \right]$$

The Riemann curvature tensor:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$$

Euler–Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0$$

Geodesic equation:

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0$$

SECTION A: *You should attempt all questions. Marks awarded are shown next to the questions.*

1. A student stands at the end of a train and throws a ball towards the front of the train. This student measures the speed of the ball to be $c/5$ and the length of the train to be 100 metres. A lecturer, standing on the station platform, observes the train to leave the station with a speed $3c/5$.

(a) What is the length of the train as measured by the lecturer?

(b) Starting from the Lorentz transformations given on page 1, calculate the speed of the ball as measured by the lecturer.

[In this question, you may assume the student and lecturer are in a standard configuration.] [12]

2. Write down the conditions for a 4-vector to be timelike and null (lightlike), respectively.

Consider two timelike 4-vectors, $\bar{A} = (A^0, A^1, 0, 0)$ and $\bar{B} = (B^0, B^1, 0, 0)$, where the components A^0, A^1, B^0, B^1 are all positive quantities. Show that the sum of the 4-vectors $\bar{A}$ and $\bar{B}$ can never be null. [10]

3. Write down the transformation law under a general coordinate transformation for a type $(1, 2)$ tensor.

Show that the product of a type $(1, 0)$ tensor with a type $(0, 2)$ tensor results in a type $(1, 2)$ tensor.

Show that the contraction of a type $(1, 2)$ tensor results in a type $(0, 1)$ tensor.

[12]

4. Consider the spacetime metric given by

$$ds^2 = -dt^2 + e^{2Ht}dx^2$$

 [This question continues overleaf ...]

where H is a constant. Calculate the covariant and contravariant components of the metric tensor for this spacetime.

Employ the formula for the connection given on page 1 to calculate the components ${\Gamma^t}_{tx}$, ${\Gamma^x}_{tx}$ and ${\Gamma^t}_{xx}$. [9]

5. The non-vanishing components of the Ricci tensor for the spacetime with metric

$$ds^2 = (1 - r^2)^{-1} dr^2 + r^2 d\theta^2$$

are given by

$$R_{rr} = \frac{1}{1 - r^2}, \qquad R_{\theta\theta} = r^2$$

Calculate the Ricci scalar for this spacetime.

Explain, briefly, why at least one of the components of the Riemann tensor for this spacetime must be non-zero. [7]

SECTION B: *Each question carries 25 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.*

6. (a) A stationary particle of rest mass M decays into two particles that both move along the x-axis. One has a rest-mass m_1 and a speed u_1 and the other has a rest-mass m_2 and a speed u_2. Prove that

$$M^2 = m_1^2 + m_2^2 + 2m_2^2 \frac{u_2}{1 - u_2^2} \left(\frac{1}{u_1} + u_2 \right)$$

(b) A stationary particle of rest mass M decays into a massive particle with rest-mass m_1 moving with speed u_1 towards the left along the x-axis and a massless particle moving in the opposite direction with energy E.

Show that

$$E = \frac{m_1 u_1}{\sqrt{1 - u_1^2}}$$

and hence derive an expression relating the rest-mass M directly to m_1 and u_1.

[In this question you may set the speed of light $c = 1$].

 [Next question overleaf.]

7. The metric for a particular two-dimensional Riemannian spacetime is given by

$$ds^2 = -y^3 dx^2 + x^4 dy^2$$

Employ the geodesic equation to calculate all the components of the connection $\Gamma^a{}_{bc}$ for this metric. Hence calculate the $R^x{}_{yxy}$ component of the Riemann tensor.

8. Write down the definition of the Ricci tensor, R_{bd}, in terms of the contraction of the Riemann tensor, $R^a{}_{bcd}$. What is the definition of the Einstein tensor, G_{ab}?

In a Local Inertial Frame at a point P in spacetime, what are the values of the components of the metric tensor, g_{ab}, and the connection, $\Gamma^a{}_{bc}$?

Starting from the definition of the connection, $\Gamma^a{}_{bc}$, given in terms of the metric and its partial derivative on page 1, prove by working in a Local Inertial Frame that $\Gamma^a{}_{ac,e}$ is symmetric under an interchange of the indices c and e.

Hence, prove that $R_{bd} = R_{db}$ in all reference frames.

Finally, using the antisymmetry properties of the Riemann tensor $R^a{}_{bcd}$, prove that $G^{cd} R^a{}_{bcd} = 0$, where G^{cd} is the contravariant form of the Einstein tensor.

9. The Schwarzschild spacetime metric given in standard form is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

where G and M are constants.

By choosing a new time coordinate $\tilde{t} \equiv t + 2GM \ln |r - 2GM|$, show that a light-ray moving along a null (lightlike) geodesic with $\theta =$ constant and $\phi =$ constant satisfies the condition

$$\left(\frac{d\tilde{t}}{dr}\right)^2 \left(1 - \frac{2GM}{r}\right) - \frac{4GM}{r} \left(\frac{d\tilde{t}}{dr}\right) - \left(1 + \frac{2GM}{r}\right) = 0$$

Explain, with the help of a light-cone diagram, why a timelike particle located initially at a value of $r < 2GM$ must inevitably reach $r = 0$.

What is the physical significance of the value $r = 2GM$?

[End of examination paper.]