$\zeta(z)$: The functional equation.

This handout gives the proof of the functional equation for the zeta function, which is similar to, but more complicated than, the reflection formula for the gamma function. You do not need to remember the formula. The last part of the proof involves some estimation which is included for completeness: for this course, you are not expected to reproduce such material.

The functional equation, which provides the analytic continuation of $\zeta(z)$ to $\Re z < 1$ and much other useful information about the ζ function, is

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z) \ .$$

This can be obtained from the Hankel integral representation

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1} dt.$$

We add to the Hankel contour a large rectangle with vertices at $z = \pm R \pm (2N+1)\pi i$, forming a closed contour C.

Let

$$J(z) = \int_C \frac{t^{z-1}}{e^{-t}-1} dt.$$

The integrand has simple poles at $z = 2\pi i m$, $m = \pm 1, \ \pm 2, \ \dots$ and the horizontal sides of the rectangle pass between two poles. The integrand also has a branch point at $z = 0$ which does not lie within C.

To evaluate the integral, we need only sum the residues at the $2N$ simple poles. The residue at $z = 2\pi i m$ is (using l'Hôpital's rule) $(-1)(2\pi i m)^{z-1}$, so

$$\begin{aligned} J(z) &= -2\pi i \left(\sum_{m=1}^{N} -(2\pi i m)^{z-1} + \sum_{m=1}^{N} -(-2\pi i m)^{z-1} \right) \\ &= 2\pi i (2\pi)^{z-1} \left(e^{i\pi(z-1)/2} + e^{-i\pi(z-1)/2} \right) \sum_{m=1}^{N} \frac{1}{m^{1-z}}. \end{aligned}$$

Note that the poles are encircled in the clockwise sense, which gives rise to the extra minus signs. Taking the limit $N \to \infty$ gives

$$J(z) = 2\pi i (2\pi)^{z-1} 2 \sin(\pi z/2) \zeta(1-z)$$

provided the series converges, for which we need $\Re(1-z) > 1$, i.e. $\Re z < 0$; we will prove the formula subject to this restriction and then use analytic continuation and the identity theorem to show that the formula holds for all z (except those values for which there are singularities).

If we assume that the contributions to the integral along the sides of the large rectangle tend to zero as $N \to \infty$ and $R \to \infty$, then we have

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2)\Gamma(1-z)\zeta(1-z)$$

as required.

But what about the contribution from the sides of the rectangle? We consider the sides separately. First note that

$$\left| \int_{\text{side}} \frac{t^{z-1}}{e^{-t}-1} dt \right| \leq \int_{\text{side}} \frac{1}{|e^{-t}-1|} \left| t^{z-1} dt \right|$$

so if the denominator is bounded away from zero, i.e.

$$|e^{-t} - 1| \geq K > 0, \tag{$*$}$$

for some K on the side we are looking at, we only have to show that

$$\int_{\text{side}} \left| t^{z-1} dt \right| \to 0.$$

For the vertical side $t = R + iy$, we have[1]

$$\int_{\text{side}} \left| t^{z-1} dt \right| = \int_{-(2N+1)\pi}^{(2N+1)\pi} \frac{1}{(R^2+y^2)^{(1-z)/2}} dy$$

and there is a similar expression for the other three sides. This integral is easily shown to tend to zero as $R \to \infty$ (even in the limit $N \to \infty$ by virtue of the condition $\Re z < 0$ (there are more Rs and ys on the bottom than on the top).[2]

Now we consider the denominator, which we only have to show is bounded away from zero. On the right vertical side, we have $t = R + iy$ with $-(2N+1)\pi \leq y \leq (2N+1)\pi$. Thus

$$|e^{-t} - 1| \geq 1 - e^{-R}$$

which certainly satisfies ($*$) for large R. On the left vertical side, we have $t = -R + iy$ so the denominator is exponentially large as $R \to \infty$, which is more than we require.

On the horizontal sides, we have $t = x \pm (2N+1)\pi i$ with $-R \leq x \leq R$. Thus

$$|e^{-t} - 1| \geq 1 + e^{-x}$$

and again ($*$) is satisfied.

Thus all is secure.

[1] In the following inequality, it is assumed that z is real: there is a little more straightforward work to do if this is not so.

[2] Actually, we can evaluate the integral explicitly; or, rather, www.integrals.com can. For the indefinite integral, it gives

$$\frac{x}{(R^2+x^2)^k} \left(1 + \frac{x^2}{R^2}\right)^k F(\tfrac{1}{2}, k, \tfrac{3}{2}, -\tfrac{x^2}{R^2})$$

where $k = (1-z)/2$ and F is the hypergeometric function (see on). We can therefore take the limits explicitly, in whichever order we like, using $F(\frac{1}{2}, k, \frac{3}{2}, 0) = 1$ or $F(\frac{1}{2}, k, \frac{3}{2}, z) \approx z^{-1/2}$ for large z and $\Re k > \frac{1}{2}$.