The second solution using the Wronskian

This is partly to remind you about the Wronskian and partly to give you a little more intuition about the origin of the log term that is usually needed in the general solution of a second-order linear ODE if the roots of the indicial equation differ by an integer. For the purposes of this course, you do not need to learn the details.

We will investigate the second solution, using the Wronskian and the first solution, of the differential equation

$$w'' + p(z)w' + q(z)w = 0 \qquad (*)$$

near a regular singular point at which the exponents (solutions of the indicial equation) differ by an integer.

Recall that the Wronskian of two functions $u(z)$ and $v(z)$ is defined by

$$\mathcal{W}[u(z), v(z)] \equiv u(z)v'(z) - u'(z)v(z), \qquad (1)$$

so that

$$\left(\frac{v}{u}\right)' \equiv \frac{uv' - vu'}{u^2} = \frac{\mathcal{W}}{u^2}. \qquad (2)$$

Recall further that, if $u(z)$ and $v(z)$ are linearly dependent, then $\mathcal{W}[u(z), v(z)] = 0$. Conversely in any region of the complex plane where u and v are non-zero, the equation $\mathcal{W}[u(z), v(z)] = 0$ can be integrated to show that $u(z)$ and $v(z)$ are linearly dependent. For analytic functions, analytic continuation can be used to extend this result to any region.

If $u(z)$ and $v(z)$ satisfy $(*)$, then

$$\mathcal{W}' = -p(z)\mathcal{W}$$

as can easily be verified by differentiating equation (1) and eliminating the second derivatives using $(*)$. Thus

$$\mathcal{W}(z) = \mathcal{W}(0) \exp\left(-\int_0^z p(t)dt\right) \qquad (3)$$

On handout 3.1, it was shown (by brute force) that, in the neighbourhood of a regular singular point, equation $(*)$ has at least one solution of the form

$$u(z) = z^{\sigma_1} \sum_0^\infty a_n z^n \equiv z^{\sigma_1} f(z),$$

where σ_1 is the exponent (root of the indicial equation) with the larger real part, and $f(z)$ is an analytic function with $f(0) \neq 0$.

Let $v(z)$ be the second solution which we are trying to find and assume that $\sigma_1 = \sigma_2 + N$, where N is a non-negative integer.

By the definition of a regular singular point, we can write $p(z)$ in the form

$$p(z) = \frac{p_{-1}}{z} + \sum_{n=0}^{\infty} p_n z^n.$$

Substituting the series for $p(z)$ into equation (**??**) and integrating term by term gives

$$\mathcal{W}(z) = A \exp\big[-p_{-1}\log z - \textstyle\sum_0^\infty (n+1)^{-1} p_n z^{n+1}\big] \equiv A z^{-p_{-1}} G_1(z),$$

where $G_1(z)$ is a function which is analytic at $z = 0$ and which could, in principle, be determined (for example, by expanding the exponential). Then

$$\begin{aligned}
(v/u)' &= \mathcal{W}[u,v]u^{-2} && \text{from (??)}\\
&= Az^{-p_{-1}}G_1(z)z^{-2\sigma_1}G_2(z) && \text{where } G_2(z) = 1/(f(z))^2 \text{ and is analytic since } f(0)\neq 0\\
&= z^{-p_{-1}-2\sigma_1}G_3(z) && \text{where } G_3(z) = AG_1(z)G(z)\\
&= z^{-\sigma_1+\sigma_2-1}G_3(z) && \text{since } \sigma_1+\sigma_2 = 1-p_{-1}\\
&= z^{-N-1}G_3(z) && \text{where } N = \sigma_1-\sigma_2\\
&= z^{-N-1}\sum_0^\infty c_n z^n && \text{for some } c_n.
\end{aligned}$$

We have used the notation that $G_i(z)$ $(i = 1, 2\ldots)$ stand for functions analytic at $z = 0$ which could in principle be determined. The relationship $\sigma_1 + \sigma_2 = 1 - p_{-1}$ comes from the indicial equation. We needed $a_0 \neq 0$ because otherwise u^{-2} would have a pole at $z = 0$. Integrating the final series gives

$$\begin{aligned}
\frac{v}{u} &= c_{\scriptscriptstyle N}\log z + z^{-N}G_4(z) + B\\
v(z) &= c_{\scriptscriptstyle N}u(z)\log z + z^{\sigma_1-N}G_5(z) + Bu(z)\\
&= c_{\scriptscriptstyle N}u(z)\log z + z^{\sigma_2}\sum_0^\infty b_n z^n + Bu(z), \qquad (4)
\end{aligned}$$

for some b_n, which is the general solution. If it happens that $c_{\scriptscriptstyle N} = 0$, then the log term is not required despite the fact that the exponents differ by an integer.

Remark If one solution of the differential equation is known in closed form (i.e. not as a series) then using the Wronskian to determine the second solution is quite effective. The method becomes decidedly less attractive if one has to use series expansions. The general formula (**??**) is not of much practical use, since there is no easy formula for the coefficients b_n.