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Wielandt's theorem

This very striking result theorem is a uniqueness theorem for the Gamma function: it says roughly that the $\Gamma(z)$ *is the only function that satisfies the recurrence relationship* $\Gamma(z+1) = z\Gamma(z)$ *and is bounded in the strip* $0 < \mathrm{Re}\, z < 1$. *The key step of the proof uses Liouville's theorem but there is some fiddling about to do first. Compared with every other result in this course, it is modern, having first seen the light of day in 1939. It is not the only uniqueness theorem for* Γ*: there is also the Bohr-Mollerup theorem for which the extra condition beyond the recurrence relation is the convexity of* $\log\Gamma(x)$*; and various other theorems which require certain asymptotic behaviour.*

The proof is not only interesting and very neat; it uses some of the core ideas of this course. It is important to understand the proof, but not to be able to reproduce it.

Wielandt's theorem

If $F(z)$ satisfies:

(i) $F(z)$ is analytic for $\mathrm{Re}\ z > 0$;

(ii) $F(z+1) = zF(z)$;

(iii) $F(z)$ is bounded in the strip $1 \le \mathrm{Re}\, z \le 2$;

(iv) $F(1) = 1$;

then $F(z) = \Gamma(z)$.

Comment: if, instead of (iv), $F(z) = k$, then $k^{-1}F(z)$ satisfies the above conditions (the recurrence relation being linear) so $F(z) = k\Gamma(z)$.

Proof

We proceed by means of two lemmas.

Lemma 1

Let $F(z)$ be any function satisfying (i) to (iv) above. Then the function $f(z)$ defined by $f(z) = F(z) - \Gamma(z)$ is entire.

Note first that conditions (i) and (ii) of the theorem show that $F(z)$ can be analytically (meromorphically) continued to $\mathrm{Re}\ z \le 0$ by the method used for $\Gamma(z)$:

$$F(z) = \frac{F(z+N)}{z(z+1)\cdots(z+N-1)} \qquad \mathrm{Re}\ z > -N.$$

Thus $f(z)$ is defined for all z, except for possible simple poles at $z = 0,\ -1,\ -2,\ \ldots$.

However, condition (iv) of the theorem shows that the residues of $F(z)$ at these points, namely

$$\frac{F(1)(-1)^n}{n!} \qquad n = 0,\ -1,\ -2,\ \ldots,$$

are the same as those of $\Gamma(z)$. The difference function $f(z)$ is therefore analytic at these points (think of the Laurent expansion), and hence entire.

Lemma 2
The difference function $f(z)$ defined in Lemma 1 is bounded in the strip $0 \leq \mathrm{Re}\, z \leq 1$.

To prove this, we establish boundedness in the adjacent strip $1 \leq \mathrm{Re}\, z \leq 2$ and then use the recurrence relation.

Since $f(z)$ is analytic, our only worry is the behaviour as $\mathrm{Im}\, z \to \pm\infty$. Since $F(z)$ is bounded by condition (iii) of the theorem, we can confine our attention to $\Gamma(z)$. We have

$$\begin{aligned} |\Gamma(z)| &= \left| \int_0^\infty e^{-t} t^{z-1} dt \right| \\ &\leq \int_0^\infty e^{-t} t^{x-1} dt \\ &\leq \int_0^\infty e^{-t} t^{2-1} dt = 1 \end{aligned}$$

since $x \leq 2$ in the strip, so $\Gamma(z)$ and hence $f(z)$ is bounded in the strip.

The result immediately transfers to the strip $0 \leq \mathrm{Re}\, z \leq 1$ using $f(z) = z^{-1} f(z+1)$, the apparent problem at $z = 0$ being solved by remembering that $f(1) = 0$ by conditions (ii) and (iv) of the theorem.

Proof of the theorem

Let $s(z) = f(z) f(1-z)$, which is analytic by lemma 1 and bounded in the strip $0 \leq \mathrm{Re}\, z \leq 1$ by lemma 2 (note that $f(1-z)$ takes the same set of values as $f(z)$ in the strip).

Now using the recurrence relation twice gives

$$s(z+1) = f(z+1) f(-z) = z f(z)\, (-z)^{-1} f(1-z) = -s(z)$$

so $s(z)$ (being periodic with period 2 and bounded in $0 \leq \mathbf{Re} z \leq 2$) is bounded for all z, and entire. It is therefore constant by Liouville's theorem.

Finally, from $s(1) = 0$, we infer that $s(z) \equiv 0$. Hence $f(z) = 0$[1] and $F(z) = \Gamma(z)$.

[1]If g and h are entire and $g(z)h(z) = 0$ then $g(z)$ or $h(z)$ is identically zero. How could it be otherwise? But if you want to prove it, you find a point z_0 for which $g(z_0) \neq 0$ and expand g and h in Taylor series about that point, multiply them together and equate coefficients of $(z - z_0)^n$ to zero.