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Example: Waves on a finite string

This is an example of the method of solving partial differential equations by integral transforms. You have to decide whether to use Laplace or Fourier transforms, and with respect to which variable. An initial value problem, such as this one, suggests Laplace transforms with respect to t, though equally a boundary value problem with finite boundaries suggests a Fourier series approach.

The two methods of inversion discussed here are of course equivalent but lead to different interpretations. In the second method, there is an interesting point concerning the time at which the solution 'switches on', determined by exponential behaviour of the Laplace transform.

A string undergoing small transverse oscillations with displacement $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

We will solve the problem where the string is fixed at one end ($x = 0$) but movable at the other ($x = l$). It is intially at rest and the movable end is given a nudge at $t = 0$ and held. Thus

$$y(0,t) = 0 \qquad y(l,t) = y_\circ \qquad (t > 0)$$

and

$$y(x,0) = \frac{\partial y}{\partial t} = 0 \qquad (0 \le x < l)$$

Alternatively, $y(x,t)$ could be the potential between the plates a parallel-plate capacitor, one plate of which is earthed while the other plate is raised and held at potential y_0.

Taking the Laplace transform of the equation with respect to t gives an ordinary differential equation in x:

$$p^2 \widehat{y}(x,p) = c^2 \frac{\partial^2 \widehat{y}}{\partial x^2}$$

with solution

$$\widehat{y}(x,p) = A(p)\sinh(px/c) + B(p)\cosh(px/c)$$

where A and B are arbitrary 'constants' (functions of p but not of x) of integration. Taking the Laplace transform of the boundary conditions gives

$$\widehat{y}(0,p) = 0 \Longrightarrow B(p) = 0$$

$$\widehat{y}(l,p) = \int_0^\infty y_\circ e^{-pt} dt = \frac{y_\circ}{p} \Longrightarrow A(p)\sinh lp/c = \frac{y_\circ}{p}.$$

Thus

$$\widehat{y}(x,p) = \frac{y_\circ}{p}\frac{\sinh(xp/c)}{\sinh(lp/c)}$$

and

$$y(x,t) = \frac{y_\circ}{2\pi i}\int_\gamma \frac{\sinh(xp/c)}{\sinh(lp/c)}\frac{e^{pt}}{p}\, dp.$$

Inversion

Method 1
Writing out the sinh functions in terms of exponentials gives

$$\widehat{y}(x,p) = \frac{y_\circ}{p}\left(e^{xp/c} - e^{-xp/c}\right)\frac{\left(1-e^{-2pl/c}\right)^{-1}}{e^{lp/c}},$$

and expanding the inverse power binomially gives

$$\frac{y_\circ}{p}\left[e^{p(x-l)/c} - e^{-p(x+l)/c}\right]\left[1+e^{-2pl/c}+e^{-4pl/c}+\cdots\right].$$

Now if the two square brackets are multiplied out, each term is of the form $kp^{-1}\exp(\alpha p)$, where k and α are independent of p. The inverse transform is a Heaviside function $H(t)$ (on account of the p^{-1}), the argument of which is shifted (on account of the exponential, using the shifting theorem). Note that along the path of integration, $Re\,p>0$ so the sum converges sufficiently well for the order of integration and summation to be exchanged. The solution is therefore an infinite sum of Heaviside functions:

$$y(x,t) = y_\circ\sum_0^\infty\left[H\big(ct+x-(2n+1)l\big) - H\big(ct-x-(2n+1)l\big)\right].$$

This form of the solution can be interpreted as a linear combination of terms caused by reflections of the original pulse at the fixed ends of the string. It can also be obtained directly from the general solution of the wave equation $y=f(x-ct)+g(x+ct)$ by implementing the boundary and initial conditions to find f and g for the various ranges of their arguments.

Method 2
The integrand has simple poles at the zeros of $\sinh(lp/c)$, i.e. at $p=m\pi ci/l$ on the imaginary axis. (Note that the pole at the origin is simple.)

For $|p|\to\infty$ with $Re\,p>0$, the integrand tends to

$$\frac{1}{p}\frac{e^{xp/c}}{e^{lp/c}}e^{pt}.$$

Thus if $x/c-l/c+t<0$, the path can be closed in the right-half plane, and since the integrand has no singularities in the right-half plane $y(x,t)=0$ for $ct<l-x$.

For $|p|\to\infty$ with $Re\,p<0$, the integrand tends to

$$\frac{1}{p}\frac{e^{-xp/c}}{e^{-lp/c}}e^{pt}.$$

Thus if $-x/c+l/c+t>0$ the path can be closed in the left-half plane, and the integral evaluated from the residues at the poles on the imaginary axis. Therefore, for $ct>x-l$

$$y(x,t) = y_\circ\frac{x}{l} + 2y_\circ\sum(-1)^m\frac{\sin(m\pi x/l)}{m\pi}\cos(m\pi ct/l). \qquad (*)$$

It is curious that the ranges for which $(*)$ holds ($ct>x-l$) and the range for which the solution vanishes ($ct<l-x$) overlap. The explanation is that the Fourier series $(*)$ in fact sums to zero for $-l+x<ct<l-x$. As expected the solution only 'turns on' at the point x at time $(l-x)/c$ showing that the disturbance travels at speed c (from the end $x=l$).