Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

Example: The causal Green's function for the wave equation

In this example, we will use Fourier transforms (in three dimensions) together with Laplace transforms to find the solution of the wave equation with a source term, representing (say) an electromagnetic potential arising from a time-varying charge distribution. The solution takes the form of an integral involving a Green's function, which has the property that only contributions from times before the present time influence the solution — in other words, a causal Green's function. There are quite a lot of technical ideas in this example which may well be unfamiliar.

Problem: Solve $\Box\phi = f(\mathbf{x}, t)$, where $\Box$ is the D'Alembertian operator $\nabla^2 - c^{-2}\partial^2/\partial t^2$, subject to the initial conditions $\phi(\mathbf{x}, 0) = \dot{\phi}(\mathbf{x}, 0) = 0$ and the boundary condition $\phi(\mathbf{x}, t) \to 0$ as $|x| \to \infty$.

Our first move is to Laplace transform the equation with respect to t, assuming that the operation of taking the Laplace transform commutes with ∇^2:

$$\nabla^2 \widehat{\phi}(\mathbf{x}, p) - (p^2/c^2)\widehat{\phi}(\mathbf{x}, p) = \widehat{f}(\mathbf{x}, p)\,.$$

Next we take the Fourier transforms of this equation with respect to x, y and z consecutively, noting that, for example,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2\widehat{\phi}(\mathbf{x}, p)}{\partial x^2} e^{ik_1x}e^{ik_2y}e^{ik_3z}dxdydz &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -k_1^2\widehat{\phi}(\mathbf{x}, p)e^{ik_1x}e^{ik_2y}e^{ik_3z}dxdydz \\ &= -k_1^2\widetilde{\widehat{\phi}}(\mathbf{k}, p)\,.\end{aligned}$$

Thus

$$(-k_1^2 - k_2^2 - k_3^2 - p^2/c^2)\widetilde{\widehat{\phi}}(\mathbf{k}, p) = \widetilde{\widehat{f}}(\mathbf{k}, p) \quad \text{i.e.} \quad \widetilde{\widehat{\phi}}(\mathbf{k}, p) = \widetilde{\widehat{G}}(\mathbf{k}, p)\widetilde{\widehat{f}}(\mathbf{k}, p)$$

where

$$\widetilde{\widehat{G}}(\mathbf{k}, p) == \frac{-1}{\mathbf{k}.\mathbf{k} + p^2/c^2}\,.$$

Using the convolution theorem in all four transform variables gives

$$\phi(\mathbf{x}, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{t} G(\mathbf{x} - \boldsymbol{\xi}, t - \tau)\, f(\boldsymbol{\xi}, \tau)\, d\tau\, d\xi_1\, d\xi_2\, d\xi_3\,,$$

where $G(\mathbf{x}, t)$ is the inverse transform, with respect to all four variables, of $\widetilde{\widehat{G}}(\mathbf{k}, p)$.

Now all we have to do is to find the Green's function $G(\mathbf{x}, t)$ by inverting the four transforms. We tackle the Fourier tranforms first. It is convenient to regard k_1, k_2 and k_3 as variables in a three-dimensional space and then covert to polar coordinates, taking the polar direction in $\mathbf{k}$-space to be parallel to the fixed vector $\mathbf{x}$, so that $\mathbf{k}.\mathbf{x} = kr\cos\theta$ (writing k for $|\mathbf{k}|$). We have

(remembering to use the Jacobian to convert to polar coordinates)

$$
\begin{aligned}
\widehat{G}(\mathbf{x},p) &= \frac{1}{(2\pi)^3}\int_0^\infty\int_0^\pi\int_0^{2\pi}\frac{-e^{i\mathbf{k}.\mathbf{x}}}{k^2+p^2/c^2}k^2\sin\theta\,dk\,d\theta\,d\phi \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^\pi\frac{-e^{ikr\cos\theta}}{k^2+p^2/c^2}k^2\sin\theta\,dk\,d\theta \qquad \text{(the } \phi \text{ integral was trival)} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty\frac{e^{-ikr}-e^{ikr}}{k^2+p^2/c^2}\frac{k}{ir}\,dk \qquad \text{(having set } u=\cos\theta \text{ to do the } \theta \text{ integral)} \\
&= -\frac{1}{(2\pi)^2}\int_{-\infty}^\infty\frac{e^{ikr}}{k^2+p^2/c^2}\frac{k}{ir}dk \qquad \text{(which has simple poles at } k=\pm ip/c\text{)} \\
&= -\frac{1}{(2\pi)^2}\frac{2\pi i}{2ir}e^{-pr/c} \qquad \text{(closing in the uhp, since } r>0\text{, and using l'Hôpital)} \\
&= -\frac{1}{4\pi r}e^{-pr/c}
\end{aligned}
$$

For the residue calculation, note that Re $p \geq 0$ on the Bromwich contour so the pole in the upper half plane is at ip/c (not $-ip/c$).

Finally, we must invert the Laplace transform to obtain $G(\mathbf{x},t)$. No work is required for this: recalling that the Laplace transform of $\delta(t)$ is 1, and that the effect of multiplying a transform by an exponential is to shift the argument of the inverse, we have

$$
G(\mathbf{x},t) = -\frac{1}{4\pi r}\delta(t-r/c)\,.
$$

Thus the Green's function is zero except on the past light cone. This surprising result in fact accords with our expectations: in the case of electromagnetic radiation signals (e.g. light) we are affected only by events on our past light cone. It is one interpretation of Huygen's principal.

Interestingly, the result is not true in a two-dimensional space (or in any space of even dimensions). In two dimensions, the Jacobian for changing to polar coordinates is just k (instead of $k^2\sin\theta$), and the lower power of k in the residue calculation gives rise to an extra power of p in the denominator, and hence a step function rather than a delta function when inverted. In two dimensions, we would see not just a flash from a lighthouse, but also a long afterglow.

For completeness, we calculate the solution $\phi(\mathbf{x},t)$ using the Green's function. We have

$$
\begin{aligned}
\phi(\mathbf{x},t) &= \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\int_0^t G(\mathbf{x}-\boldsymbol{\xi},t-\tau)\,f(\boldsymbol{\xi},\tau)\,d\tau\,d\xi_1\,d\xi_2\,d\xi_3 \\
&= \frac{-1}{4\pi}\int\limits_{\text{all }\boldsymbol{\xi}-\text{space}}\int_0^t\frac{\delta(t-\tau-R/c)}{R}\,f(\boldsymbol{\xi},\tau)\,d\tau\,d^3\xi \qquad \text{(writing } R \text{ for } |\mathbf{x}-\boldsymbol{\xi}|\text{)} \\
&= \frac{-1}{4\pi}\int\limits_{\text{all }\boldsymbol{\xi}-\text{space}}\frac{f(\boldsymbol{\xi},t-R/c)}{R}\,d^3\xi
\end{aligned}
$$

and we see again that only effects from the past light cone contribute.