Mathematical Tripos Part II — Michaelmas term 2007

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Product formulae for the Gamma function

The purpose of this handout is to give an alternative classical approach to the Gamma function, defining it and deducing its properties from a product formula rather than an integral. You you do not need to learn these formulae or the details of the derivations.

The gamma function can be defined in terms of an infinite product, the *Euler product formula*:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}.$$

For $Re\ z > 0$, this formula can be obtained from the Euler integral

$$\int_0^\infty e^{-t} t^{z-1} dt$$

by writing e^{-t} in the standard limiting form

$$e^{-t} = \lim_{n\to\infty} (1 - t/n)^n$$

and then writing the infinite integral as the limit of a finite integral

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n t^{z-1}(1 - t/n)^n \, \mathrm{d}t.$$

Changing variable to $\tau = t/n$ gives

$$\Gamma(z) = \lim_{n\to\infty} n^z \int_0^1 \tau^{z-1}(1-\tau)^n \, \mathrm{d}t = \lim_{n\to\infty} n^z \int_0^1 \frac{n}{z}\, \tau^z (1-\tau)^{n-1} \, \mathrm{d}t = \cdots$$

and integrating n times by parts and then integrating once more gives the product formula. Some fiddly justification is needed for the interchange of order of the various limiting processes.

The Euler product formula can be written without the limit as

$$\Gamma(z) = \frac{1}{z} \prod_{m=1}^{\infty} \left[\left(1 + \frac{1}{m}\right)^z \left(1 + \frac{z}{m}\right)^{-1} \right].$$

It can be seen that this is essentially the Euler product a follows. First rewrite the fractions in the brackets:

$$\Gamma(z) = \frac{1}{z} \prod_{m=1}^{\infty} \left[\left(\frac{m+1}{m}\right)^z \left(\frac{m}{m+z}\right) \right].$$

Now write out the first $n-1$ terms in the product, noting that almost all of the terms from the first bracket cancel:

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} \frac{1}{z} \left[\left(\frac{n}{n-1}\, \frac{n-1}{n-2} \cdots \right)^z \left(\frac{n-1}{n-1+z}\, \frac{n-2}{n-2-z} \cdots \right) \right] \\ &= \lim_{n\to\infty} \frac{1}{z} \left[n^z \left(\frac{n-1}{n-1+z}\, \frac{n-2}{n-2-z} \cdots \frac{1}{1+z} \right) \right]\end{aligned}$$

and this last expression is more or less the Euler product formula. (There is a factor of $n/(n+z)$ missing, but this factor is approximately 1 for large n.)

The infinite product converges provided $z \neq 0, -1, -2, \ldots$ (see Copson, p209) so this formula provides the meromorphic continuation of $\Gamma(z)$ to the whole of $\mathcal{C}$. It is apparent from the product formula that $\Gamma(z)$ is single-valued and has simple poles at non-positive integers.

A further definition of the gamma function, used by Weierstrass, is the *canonical product* of the Hadamard form:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} (1 + z/k)e^{-z/k}.$$

Here, γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right).$$

This follows directly from the previous infinite product expression on slipping the definition of γ under the product sign as follows. Starting from the reciprocal of the Euler formula, we have:

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z \lim_{n\to\infty} \frac{(1+z)(2+z)\cdots(n+z)}{n!n^z} = z \lim_{n\to\infty} e^{-z\log n}(1+z)(1+z/2)\cdots(1+z/n) \\
&= z \lim_{n\to\infty} e^{-z[\log n - (1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})]} e^{-z[1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{+}{1}n]}(1+z)(1+z/2)\cdots(1+z/n) \\
&= ze^{\gamma z} \prod_{k=1}^{\infty} (1+z/k)e^{-z/k}
\end{aligned}$$

The exponential factor $e^{-z/k}$ in this definition ensures that the infinite product converges comfortably.

It can immediately be seen from the canonical product that $1/\Gamma(z)$ is an entire function, with simple zeros at $z = 0, -1, \ldots,$ so $\Gamma(z)$ is holomorphic except for simple poles at these points, and has no zeros.

Finally, since there seems to be some space left, here is a sketch proof that the Euler-Mascheroni constant exists. From a graph of $1/x$ we see that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^n \frac{1}{x} dx < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

i.e.

$$S_n - 1 < \log n < S_n - \frac{1}{n}$$

where $S_n = \sum_1^n \frac{1}{n}$. Thus

$$\frac{1}{n} < S_n - \log n < 1.$$

The difference between the sum and the integral increases with n, and so tends to a limit. The value of γ is about 0.577; it is not known whether it is rational though it is suspected that it is transcendental.