Mathematical Tripos Part II — Michaelmas term 2007
Further Complex Methods — Dr S.T.C. Siklos

The Papperitz equation

In the lectures, I showed that there is only one differential equation that has exactly one singular point, which is regular (remember that the point at infinity is to be included). This handout gives the corresponding result for three regular singular points. You need to understand the calculation, but the derivation of formula for $q(z)$ is a matter of not very interesting algebra on which it would be unrewarding to spend much time.

The most general second-order linear ordinary differential equation with exactly three singular points, all of which are regular, is called the *Papperitz equation*. Miraculously, it is completely determined by the position of the three singular points and the pairs of exponents at each point — a total of 8 complex parameters (not 9, since the six exponents must sum to 1).

Even more miraculous is the fact that the complete solutions near any of the singular points can be expressed in terms of just one function, the *hypergeometric function*, $F(a, b; c; z)$, which depends on just three parameters. The extra five parameters are introduced by scaling of the dependent variable (w) (two parameters, which changes the exponents) and Möbius transformation of the independent variable (z) (three parameters, which changes the position of the singularities).

It may seem arcane to study a differential equation constrained by what appear to be very restrictive conditions, but in fact the Papperitz equation crops up very often in all branches of mathematics, and many of the commonly used special functions (Bessel, Airy, Legendre and many more) can be derived from solutions of the Papperitz equation.

Suppose the only singular points of the differential equation $w'' + p(z)w' + q(z)w = 0$ are at $z = a$, $z = b$ and $z = c$, all regular singular points. All other points are ordinary, including the point $z = \infty$. That means that $p(z)$ has (at most) simple poles at $z = a$, $z = b$ and $z = c$ and $q(z)$ has (at most) double poles at these points. Furthermore, $p(z) = 2z^{-1} + O(z^{-2})$ and $q(z) = O(z^{-4})$ as $z \to \infty$.

We may write, without loss of generality,

$$p(z) = \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} + P(z)$$

where $P(z)$ is entire. The residues at the poles are expressed in terms of six constants α, α', ..., for reasons that will emerge; but clearly only the combinations $\alpha + \alpha'$, etc, are determined by $p(z)$.

Since $p(z) \to 0$ as $z \to \infty$, $P(z)$ must also tend to zero and is therefore identically zero by Liouville's theorem. Further, $zp(z) \to 2$, so $3 - \alpha - \alpha' - \beta - \beta' - \gamma - \gamma' = 2$ whence the important result that the six exponents at the regular singular points (as they will turn out to be) sum to 1.

The obvious way to write the general form of $q(z)$, having three double poles is

$$q(z) = \frac{k_a z + k'_a}{(z-a)^2} + \frac{k_b z + k'_b}{(z-b)^2} + \frac{k_c z + k'_c}{(z-c)^2} + \text{ an entire function .}$$

If we now add these terms, we obtain an expression of the form

$$q(z) = \frac{Q_1(z)}{(z-a)^2(z-b)^2(z-c)^2} + Q_2(z)$$

where $Q_1(z)$ is a polynomial of degree 5 in z and $Q_2(z)$ is an entire function. However, the condition that $z^4 q(z)$ is bounded as $z \to \infty$ (it is analytic in z^{-1}) shows first that $Q_2(z) \equiv 0$ by Liouville's theorem (using just $q(z) \to 0$ as $z \to 0$) and then that $Q_1(z)$ can be at most a quadratic function of z.

That means we can write $q(z)$ without loss of generality in the form

$$q(z) = \frac{1}{(z-a)(z-b)(z-c)} \left(\frac{q_a}{z-a} + \frac{q_b}{z-b} + \frac{q_c}{z-c} \right)$$

where q_a, q_b and q_c are constants.

Finally, we write (again without loss of generality, since we are assuming that a, b and c are distinct) $q_a = \alpha\alpha'(a-b)(a-c)$, with similar cyclic expressions for q_b and q_c to obtain the Papperitz equation:

$$w'' + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right) w' \\ - \frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)} \left(\frac{\alpha\alpha'}{(z-a)(b-c)} + \frac{\beta\beta'}{(z-b)(c-a)} + \frac{\gamma\gamma'}{(z-c)(a-b)} \right) w = 0.$$

We can see the significance of the Greek constants in this equation, and the reason for the elaborate choices of constants, by approximating the equation in the neighbourhood of one of the singular points. For $z \approx a$, we have

$$w'' + \frac{1-\alpha-\alpha'}{z-a} w' + \frac{\alpha\alpha'}{(z-a)^2} w \approx 0$$

so the exponents at this regular singular point are α and α'.

The form of this equation is preserved under Möbius transformations of the independent variable. We can check this by decomposing the general such transformation into translations, (complex) scalings and inversion. Clearly, the only effect of the translation $z \to z + k$ is to change the positions of the singular points (they remain regular, no further singular points are introduced and the exponents are unchanged). The same is true for the scaling $z \to kz$: this can be seen almost by inspection.

For the inversion

$$t = \frac{1}{z}$$

we have

$$\frac{d}{dz} = -t^2 \frac{d}{dt}; \qquad \frac{d^2}{dz^2} = 2t^3 \frac{d}{dt} + t^4 \frac{d^2}{dt^2}$$

so the equation transforms to

$$0 = \ddot{w} + \frac{1}{t} \left[2 - \frac{1-\alpha-\alpha'}{1-at} - \frac{1-\beta-\beta'}{1-bt} - \frac{1-\gamma-\gamma'}{1-ct} \right] \dot{w} \\ - \frac{(1/b-1/c)(1/c-1/a)(1/b-1/a)}{(t-1/a)(t-1/b)(t-1/c)} \times \\ \left[\frac{\alpha\alpha'}{(t-1/a)(1/b-1/c)} + \frac{\beta\beta'}{(t-1/b)(1/c-1/a)} + \frac{\gamma\gamma'}{(t-1/c)(1/c-1/a)} \right] w.$$

The coefficient of w is exactly the correct form for a Papperitz equation with singular points at $t = 1/a$, $1/b$ and $1/c$ with exponents α, α', etc.

The coefficient of $\dot{w}$ is also of the correct form, though it doesn't look it. But setting

$$\frac{1}{1-at} = 1 + \frac{at}{1-at},$$

gives for the coefficient

$$\frac{1}{t}\left[2-(1-\alpha-\alpha')(1+\frac{t}{1/a-t})-(1-\beta-\beta')(1+\frac{t}{1/b-t})-(1-\gamma-\gamma')(1+\frac{t}{1/c-t})\right]$$

The constant terms in the big square brackets sum to zero using the fact that the six exponents sum to 1 which gives the

$$\frac{1-\alpha-\alpha'}{t-1/a}$$

(etc) terms that we are looking for.