Mathematical Tripos Part II | Michaelmas term 2007
Further Complex Methods | Dr S.T.C. Siklos

Example: Legendre's equation

In this hand out, Legendre's question will be written as a Papperitz equation, the solutions of which can be written down in terms of hypergeometric functions with no further calculation.

Legendre's equation comes from solving Laplace's equation $\nabla^2\phi = 0$ in spherical polar coordinates by separation of variables using spherical coordinates. It is the equation corresponding to the θ variable.

With $z = \cos\theta$, we have

$$(1-z^2)w'' - 2zw' + \left(n(n+1) - \frac{m^2}{1-z^2}\right)w = 0.$$

The parameter m arises as a separation constant connected with the ϕ equation (an integer, so that the solution is single-valued in ϕ) and the parameter n is related to the separation constant connected to the θ equation, which must of of the form $n(n+1)$ to prevent singularities along the axes $\theta = 0$ and $\theta = \pi$.

In the standard notation,

$$p(z) = \frac{-2z}{1-z^2}, \qquad q(z) = \frac{n(n+1)}{1-z^2} - \frac{m^2}{(1-z^2)^2}.$$

There are regular singular points at $z = \pm 1$, since $(1 \pm z)p(z)$ and $(1 \pm z)^2 q(z)$ are analytic at $z = \pm 1$. The only other singular point is $z = \infty$, which is also regular, since $zp(z)$ and $z^2 q(z)$ are analytic functions of z^{-1} for large z. Thus the Legendre equation has exactly three singular points, all regular, and is a Papperitz equation. We can therefore determine the solution as a P-function and hence as a linear combination of hypergeometric functions.

We could easily rewrite the Legendre equation in the standard form for the Papperitz equation with singularities at ± 1 and ∞ (if we could remember it) in order to identify the parameters in the P-function. However, it is just as quick to approximate the equation near the singular points to find the exponents, which are exactly the required parameters.

Near $z = 1$,

$$p(z) \approx \frac{1}{(z-1)}, \qquad q(z) \approx \frac{-m^2}{4(z-1)^2},$$

so the indicial equation is

$$\sigma(\sigma - 1) + \sigma - \tfrac{1}{4}m^2 = 0$$

and the exponents at $z = 1$ (and at $z = -1$, by symmetry) are $\pm m/2$.

Near $z = \infty$,

$$p(z) \approx \frac{2}{z}, \qquad q(z) \approx -\frac{n(n+1)}{z^2}.$$

Setting, $w = (1/z)^\sigma$ shows that the indicial equation is

$$-\sigma(-\sigma - 1) - 2\sigma - n(n+1) = 0$$

so the exponents at $z = \infty$ are $\sigma = -n$ and $\sigma = n + 1$.

The Legendre equation therefore corresponds to the P-function

$$P\left\{\begin{array}{cccc} 1 & \infty & -1 & \\ m/2 & -n & -m/2 & z \\ -m/2 & n+1 & m/2 & \end{array}\right\}.$$

But

$$P\left\{\begin{array}{cccc} 1 & \infty & -1 & \\ -m/2 & -n & m/2 & z \\ m/2 & n+1 & -m/2 & \end{array}\right\} = (1-z)^{-m/2}(1+z)^{m/2}P\left\{\begin{array}{cccc} 1 & \infty & -1 & \\ 0 & -n & 0 & z \\ m & n+1 & -m & \end{array}\right\}$$

$$= \left(\frac{1+z}{1-z}\right)^{m/2} P\left\{\begin{array}{cccc} 0 & \infty & 1 & \\ 0 & -n & 0 & (1-z)/2 \\ m & n+1 & -m & \end{array}\right\}$$

In terms of the standard hypergeometric function parameters, the last P-function has $m = 1-c,\ \ -n = a,\ \ n+1 = b.$

Thus two linearly independent solutions of Legendre's equation can be written, in the case when m is not an integer (so that the exponents at the origin do not differ by an integer), as

$$\left(\frac{1+z}{1-z}\right)^{m/2} F(-n, n+1; 1-m; \tfrac{1-z}{2})$$

and[1]

$$(1-z^2)^{m/2}F(-n+m, n+m+1; m; \tfrac{1-z}{2}).$$

Of course, as remarked above, m is an integer in physical situations. In this case, if $m \le 0$ (as we may choose without loss of generality), the first solution above, corresponding to the larger exponent, is still valid, but the second solution above may need to be replaced by a solution of the log form.

If, in addition, n is also an integer (as is again the physical requirement), then the series for the hypergeometric function $F(-n, n+1, 1-m, \frac{1-z}{2})$ will terminate giving a a polynomial.[2]

The second solution above would not need to be replaced by a log term if the series terminated before it blew up, i.e. if $-n+m > m$ or $n+m+1 < m$, one of which will necessarily hold (unless $n = 0$ or $n = -1$ both of which give a constant polynomial). In this case, the second solution is therefore not logarithmic.

[1]Remember that the second solution, using the standard parameters, is $z^{1-c}F(a+1-c, b+1-c; 2-c; z)$.

[2]Recall that

$$F(a, b; c; 1) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots.$$