# Solutions of the hypergeometric equation

*In the handout, the symmetries of the Riemann $P$-function are used to derive the second solution of the hypergeometric equation near the origin and also the two solutions near $z = \infty$ in terms of hypergeometric functions. This just scratches at the surface of the mine of symmetries of the $P$-function.*

## The second solution

The principle branch of the $P$-function

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}$$

corresponding to the exponent 0 at $z = 0$ is the hypergeometric function $F(a, b; c; z)$. It turns out, delightfully, that the principle branch corresponding to the exponent $1-c$ can be expressed in terms of $F$, and that, even more delightfully, no calculation at all is required to do so.

Note first that $w(z)$, the branch we are seeking, is of the form $w(z) = z^{1-c}g(z)$, where $g(z)$ is analytic at 0 and $g(0) = 1$.

Now $w(z)$ is a branch of the hypergeomtric $P$-function

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}$$

so, by shifting, $z^{c-1}w(z)$ is a branch of

$$\begin{aligned} P\begin{Bmatrix} 0 & \infty & 1 & \\ c-1 & a-c+1 & 0 & z \\ 0 & b-c+1 & c-a-b & \end{Bmatrix} &= P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a-c+1 & 0 & z \\ c-1 & b-c+1 & c-a-b & \end{Bmatrix} \\ &\equiv P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a' & 0 & z \\ 1-c' & b' & c'-a'-b' & \end{Bmatrix}, \end{aligned}$$

(the second equality is trivial: the order we write the two exponents at a given singular point makes no difference to the $P$-function), where

$$a' = a-c+1; \qquad b' = b-c+1; \qquad 1-c' = c-1 \text{ ( i.e. } c' = 2-c)$$

Thus $z^{c-1}w(z)$ is a branch of the hypergeomtric $P$-function with parameters $a'$, $b'$ and $c'$. Furthermore, we know that $z^{c-1}w(z)$ is analytic at $z = 0$, so

$$z^{c-1}w(z) = F(a', b'; c'; z).$$

The second principle branch of the hypergeomtric $P$ function, and hence the second solution of the hypergeometric equation, near $z = 0$ is therefore

$$z^{1-c}F(a-c+1, b-c+1, 2-c; z).$$

Note that, as expected, this is symmetric in $a$ and $b$.

### Solutions near $z = \infty$

The two principal branches at $z = \infty$ of a hypergeometric $P$-function can be written in terms of hypergeometric functions as follows. Note first that the branches are of the form

$$P_a(z) = (1/z)^a g_a(z) \qquad \text{and} \qquad P_b(z) = (1/z)^b g_b(z)$$

where $g_a(t^{-1})$ and $g_b(t^{-1})$ are analytic at $t = 0$.

Now $P_a(z)$ is a branch of the hypergeomtric $P$-function

$$P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

so (by exponent shifting) $g_a(z)$ is a branch of

$$\begin{aligned} & P\left\{\begin{matrix} 0 & \infty & 1 & \\ a & 0 & 0 & z \\ 1-c+a & b-a & c-a-b & \end{matrix}\right\} \\ &= P\left\{\begin{matrix} \infty & 0 & 1 & \\ a & 0 & 0 & z^{-1} \\ 1-c+a & b-a & c-a-b & \end{matrix}\right\} && \text{by Möbius transformation} \\ &= P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z^{-1} \\ b-a & 1-c+a & c-a-b & \end{matrix}\right\} && \text{reordering columns} \\ &\equiv P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a' & 0 & z^{-1} \\ 1-c' & b' & c'-a'-b' & \end{matrix}\right\} \end{aligned}$$

where $c' = 1 + a - b$, $b' = 1 - c + a$ and $a' = a$.

Now $g_a(z)$ is analytic at $z^{-1} = 0$ and $g(\infty) = 1$ so $g_a(z)$ must be the principle branch of the above $P$-function corresponding to the exponent 0 at the point $z^{-1} = 0$, which by definition is a hypergeometric function.

Thus

$$w(z) = z^{-a}F(a', b'; c'; z^{-1}) = z^{-a}F(a, 1-a+c; 1+a-b; z^{-1}).$$

The other branch is obtained from this by interchanging $a$ and $b$.

Note that since there are only two linearly independent branches at each point, we can express the analytic continuation of $F(a, b; c; z)$ to large $z$ in the form

$$F(a, b; c; z) = Az^{-a}F(a, 1-a+c; 1+a-b; z^{-1}) + Bz^{-b}F(b, 1-b+c; 1+b-a; z^{-1}),$$

where $A$ and $B$ are constants which can be found using, for example, integral representation of $F(a, b; c; z)$.